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Asylum Assignment and Burden-Sharing

Gian Caspari
Leibniz Centre for
European Economic Research

Manshu Khanna
Peking University

December 2024

Working Paper 20241203

Abstract

We analyze the problem of matching asylum seekers to member states, incorporating wait times, preferences of asylum seekers, and the priorities, capacities, and burden-sharing commitments of member states. We identify a unique choice rule that addresses feasibility while balancing priorities and capacities. We examine the effects of both homogeneous and heterogeneous burden-sizes among asylum seekers on the matching process. Our main result shows that when all asylum seekers are treated as having identical burden-sizes, the asylum-seeker-proposing cumulative offer mechanism guarantees both stability and strategy-proofness. In contrast, when burden-sizes vary, there are scenarios where achieving stability or strategy-proofness is no longer possible.

Keywords: matching theory, market design, stability, asylum assignment
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Peking University HSBC Business School
University Town, Nanshan District
Shenzhen 518055, China

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Gian Caspari† Manshu Khanna‡

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We analyze the problem of matching asylum seekers to member states, incorporating wait times, preferences of asylum seekers, and the priorities, capacities, and burden-sharing commitments of member states. We identify a unique choice rule that addresses feasibility while balancing priorities and capacities. We examine the effects of both homogeneous and heterogeneous burden-sizes among asylum seekers on the matching process. Our main result shows that when all asylum seekers are treated as having identical burden-sizes, the asylum-seeker-proposing cumulative offer mechanism guarantees both stability and strategy-proofness. In contrast, when burden-sizes vary, there are scenarios where achieving stability or strategy-proofness is no longer possible.

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*This paper replaces and subsumes Caspari (2019). We are grateful to Tayfun Sönmez, M. Utku Ünver, M. Bumin Yenmez, David Delacretaz, Alex Teytelboym, and Kenzo Imamura for their valuable feedback, as well as to participants at the Boston College dissertation workshop, the Nuffield postdoc seminar, Match-Up 2019, and the SAET Conference 2019. Gian Caspari acknowledges financial support from the Swiss National Science Foundation (SNSF) through a doctoral mobility stipend, facilitating his research visit to Oxford University during this project.

†Department of Market Design, ZEW — Leibniz Centre for European Economic Research, Mannheim 68161, Germany. Email: gian.caspari@zew.de.

‡Peking University HSBC Business School, Shenzhen 518055, China. Email: manshu@phbs.pku.edu.cn.

1 Introduction

When arriving in the European Union, an asylum seeker must submit an application for protection in a single member state. If successful, the person will be granted refugee status or subsidiary protection by the country that examined the asylum claim. The responsible member state cannot be chosen freely. Under the Common European Asylum System (CEAS), the asylum seekers are required to lodge their application for protection in the country in which they initially arrive.[1] This places a disproportionate burden on countries located at the border of the European Union. Its decentralized approach to asylum assignments leads to delays and disputes over responsibility, while its strict no-choice approach incentivizes asylum seekers to engage in illegal secondary movements to reach a more preferred member state (Maiani, 2016).

1.1 An Alternative Asylum System

A centralized asylum system provides an alternative to the current decentralized approach by considering both the preferences of asylum seekers and the priorities of member states. Under this system, member states retain control over eligibility determinations, while asylum seekers still apply for asylum at a single member state. Upon registration, asylum seekers would submit their preferences, and a centralized clearinghouse would then assign responsibility to a member state based on this information. Once a responsible member state is designated, asylum seekers must travel to and remain in that state while awaiting their asylum decision.

Unlike the current system, the centralized approach allows asylum seekers to express preferences for specific member states, giving them greater agency in the process. This ensures their preferences are factored into the assignment, making the system more responsive to their needs. Meanwhile, member states maintain control over eligibility determinations, allowing them to prioritize and regulate decisions according to their own standards. Importantly, the centralized system designates a responsible state before the asylum application is processed, avoiding the logistical challenge of relocating refugees after their status is determined.

The decentralized system often struggles to process claims in a timely manner, as required by the Charter of Fundamental Rights of the European Union (Beck et al., 2014).[2]

[1]This rule is stated in Article 9 on page 62 of the *Text of the agreement on Asylum and migration management regulation*. Document last accessed on 22 October 2024 at https://www.europarl.europa.eu/meetdocs/2014_2019/plmrep/COMMITTEES/LIBE/DV/2024/02-14/06.RAMM_Asylumandmigrationmanagement_EN.pdf.

[2]The suggested timeline for a standard asylum claim is six months, and member states must inform

The centralized system addresses this by allowing asylum seekers to express preferences for member states with shorter wait times, avoiding overburdened states and reducing delays.

European regulators have also proposed burden-sharing quotas that set targets for the number of asylum applications each member state should process. As European Commission (2016) states, "A corrective allocation mechanism should be established to ensure a fair sharing of responsibility between Member States [...] in situations when a Member State faces a disproportionate number of applications for international protection." We show that a centralized mechanism is particularly well-suited to facilitate the implementation of these burden-sharing quotas across the EU.

1.2 Summary of Model and Analysis

We formulate the asylum system as matching with contracts problem (Hatfield and Milgrom, 2005), where a contract specifies an asylum seeker, a member state, and a wait time. Asylum seekers submit their preferences over combinations of wait times and member states. Following the CEAS, asylum seekers are restricted to a single application for protection, making it a many-to-one matching problem. An asylum seeker making an application can either represent an individual or a group of immediate family members applying for asylum at the same time. Therefore, each asylum seeker is given a burden-size. Member states are required to make an asylum decision after the agreed-upon wait time ends. When scheduling asylum applications, member states are constrained by their bureaucratic capacities, representing the maximum number of asylum claims that can be processed in a given period. A member state's burden-sharing quota specifies the total amount of burden-size over asylum applications that have to be scheduled. Finally, to decide between different asylum seekers, each member state (strictly) ranks asylum seekers in terms of priority.[3]

We focus on finding a *stable* and *strategy-proof* mechanism for this problem. A stable mechanism avoids situations in which an asylum seeker prefers a different contract to the one she is assigned, and the member state specified in the more desirable contract is willing to handle her claim in the specified wait time. A strategy-proof mechanism (for asylum seekers) makes sure that misreporting preferences is not beneficial for asylum seekers, and

applicants if the process takes longer. However, providing an estimated completion date does not imply a strict obligation to meet that timeframe (Beck et al., 2014).

[3]Member states' priority ranking criteria can be chosen freely as long as they don't interfere with article 14 of the European Convention on Human Rights (Jones and Teytelboym, 2017a), which prohibits discrimination on any grounds such as religion or race (European Council of Human Rights, 2013). For example, in 2015, the former prime minister of Britain, David Cameron, announced the acceptance of up to 20,000 refugees from Syria, prioritizing vulnerable children and orphans (BBC, 2015). Ties between asylum seekers with identical characteristics may be broken randomly whenever necessary.

thus it is their best interest to state their preferences truthfully.[4]

We characterize a member state choice rule over contracts in Theorem 1. This choice rule takes into account a member state's priority order, bureaucratic capacities, and burden-sharing quota, to make a selection from a given set of contracts. Given the member state choice rule, in Theorem 2 we show that: if asylum seekers are treated as having identical burden-sizes, the asylum-seeker-proposing cumulative offer mechanism (Hatfield and Milgrom, 2005) is stable and strategy-proof. However, when asylum seekers have different burden-sizes, Example 1 and Example 2 illustrate why both stability and strategy-proofness might not be achievable.

The proof of Theorem 2 builds upon the results developed in the literature on many-to-one matching markets with contracts. In particular, the member state choice rule (characterized in Theorem 1) violates the substitutes condition and the law of aggregate demand (Kelso Jr and Crawford, 1982; Hatfield and Milgrom, 2005; Hatfield and Kojima, 2010). However, for the case of homogenous burden-sizes, we show that a completion of the choice rule exists (Lemma 1), as in Hatfield and Kominers (2019), that satisfies the substitutes condition if families are prioritized over individual asylum seekers (Proposition 3). While the law of aggregate demand is satisfied if individual asylum seekers are prioritized over families (Proposition 4). The cumulative offer mechanism is stable and strategy-proof for identical burden-sizes because both requirements on the priority orderings are satisfied simultaneously.

We can obtain the same result by constructing an associated one-to-one market following Kominers and Sönmez (2016) to show that the cumulative offer mechanism is stable and strategy-proof for asylum seekers. We use Hatfield and Kominers (2019) because it helps us illustrate the lacking of cumulative offer mechanism in handling heterogeneous burden sizes (see Example 3 and Example 4).

1.3 Related Literature

The concept of managing refugee flows through organized systems has been discussed in the literature, starting with Schuck (1997), who proposed that each member state should bear a share of responsibility for temporary protection and permanent resettlement based on a quota system. Moraga and Rapoport (2014) further developed this idea, proposing a system for trading quotas multilaterally without monetary exchange, aimed at resettling longstanding refugees and asylum seekers within the European Union. They were also the first to mention the possibility of combining a quota system with a matching mechanism, paving the way for subsequent models.

[4] We consider manipulations on the part of asylum seekers only. That is, we assume that member states cannot misreport priorities.

Jones and Teytelboym (2016, 2017a,b) expanded on this by advocating for the use of matching mechanisms to enhance or replace existing resettlement practices. They differentiated between two levels of refugee matching: the global refugee match, operating at an international scale, and the local refugee match, which focuses on community-level integration within a country. In the international context, Jones and Teytelboym (2017a,b) suggested that the "thickness" of the market allows refugee allocation to be effectively modeled as a standard school choice problem (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Thus asylum seekers, whether individuals or families, can be treated equivalently in terms of burden-size, simplifying the allocation process.

The local refugee match, however, presents more complex challenges (Jones and Teytelboym, 2018). Delacrétaz et al. (2023) introduced a framework that incorporates multidimensional knapsack constraints, accounting for the thinness of the market and the limitations of local resources. Andersson et al. (2018) proposed a dynamic model that assigns refugees to localities based on types and locality-specific quotas. Additionally, Andersson and Ehlers (2020) examined a market for allocating private housing to refugees, where landlords have preferences over the sizes of refugee families and the native languages they speak, highlighting the nuanced preferences that need to be considered in local resettlement. Ahani et al. (2021) and Ahani et al. (2024) provided practical insights by assisting a U.S. resettlement agency in matching refugees to their initial placements, achieving improved employment outcomes.

Our work differs from previous studies by incorporating wait times into the global refugee match and addressing the differential burden-sizes across asylum seekers. Unlike past models, which assume the refugees' status is determined prior to matching, our approach directly matches asylum seekers, allowing member states to retain control over the eligibility determination process. This conceptual shift necessitates the use of new framework based on matching with contracts model that must account for the strategic considerations over both member states and wait times.

Our work also relates to the literature on stable matchings with sizes, where complementarities arise from a combination of bureaucratic capacities and differential burden-sizes rather than hard capacity constraints (Dean et al., 2006; McDermid and Manlove, 2010; Biró and McDermid, 2014; Yenmez, 2018; Nguyen and Vohra, 2018; Delacrétaz, 2019). While the non-existence of stable allocations in matching with sizes is often due to rigid capacity limits, instability in our problem does not arise from the same, as burden-sharing quotas are not hard constraints.

Our work is also connected to the literature on matching with contracts, which has been explored in various contexts (Aygün and Turhan, 2020; Hassidim et al., 2017; Yenmez, 2018; Sönmez and Switzer, 2013; Dimakopoulos and Heller, 2019). Among these studies, our setup is most closely related to Dimakopoulos and Heller (2019). Of particular relevance is Dimakopoulos and Heller (2019), which examines the German entry-level labor market for lawyers, incorporating wait times as a contract term. While both models address wait times, our approach uniquely accounts for member state choice rules, the consideration of differential burden-sizes, and the inclusion of bureaucratic capacities, highlighting the unique complexities inherent in asylum assignment.

2 Model and Definitions

An **asylum seeker matching problem** consists of

1. a finite set of asylum seekers A,
2. a finite set of member states M,
3. a finite set of wait times $W \subset \mathbb{R}_+$,
4. a burden-size for each asylum seeker $s : A \mapsto \mathbb{N}$,
5. a list of burden-sharing quotas $q = (q_m)_{m \in M}$ with $\sum_{m \in M} q_m \geq \sum_{a \in A} s(a)$.
6. a list of bureaucratic capacities $r = (r_m^w)_{m \in M, w \in W}$ with $\sum_{w \in W} r_m^w \geq |A|$ for all $m \in M$,
7. a list of preference rankings $P = (P_a)_{a \in A}$ over $M \times W$, and
8. a list of priority orders $\pi = (\pi_m)_{m \in M}$ over A.

An asylum seeker's **burden-size** $s(a)$ specifies the amount of burden an asylum seeker's application consumes from a member state's burden-sharing quota. This allows an asylum maker applying on behalf of her immediate family differently from an asylum seeker applying as an individual. We say that an asylum seeker matching problem $\langle s, P \rangle$ specifies the **same burden-size** for every asylum seeker if $s(a) = s(a')$ for all $a, a' \in A$.

A member state's **burden-sharing quota** q_m specifies the minimum number of applications a member state has to process. Since every asylum seeker has the right to submit an application, we require that the sum of burden-sharing quotas exceeds the sum of burden-sizes, that is, $\sum_{m \in M} q_m \geq \sum_{a \in A} s(a)$.

Each asylum seeker's application takes up one unit of **bureaucratic capacity**. A member state m can process at most $r_m^w \geq 0$ asylum seekers in time period w. Moreover, total bureaucratic capacities are sufficiently large to schedule any asylum seeker if a long enough time horizon W is considered, that is, $\sum_{w \in W} r_m^w \geq |A|$ for all $m \in M$.

Each asylum seeker has a **preference ranking** P_a over wait-time-member-state combinations $M \times W$, with the corresponding weak preference written as R_a.[5] Let $\mathscr{P}$ denote the set of all preference profiles.[6]

Each member state has a **priority order** π_m over the set of asylum seekers A, indicating the order in which they would prefer to accept applicants.

We fix A, M, W, q, r, and π, denoting an asylum seeker matching problem by $\langle s, P \rangle$. To encompass wait times, we use a matching with contracts framework. A **contract** $x = (a, m, w) \in X = A \times M \times W$ specifies an asylum seeker $a \in A$, a member state $m \in M$, and a wait time $w \in W$. Let $\mathscr{X}$ denote the set of all subsets of X. Given a contract $x \in X$, let a_x represent the asylum seeker, m_x the member state, and w_x the wait time specified in the contract. For some subset of contracts $X' \subseteq X$, let $X'_a = \{x \in X' : a_x = a\}$ denote the set of contracts asylum seeker a is part of, with equivalent notation for member states and wait times. Let $A(X') = \{a \in A : a_x = a \text{ for some } x \in X'\}$ denote the set of asylum seekers specified in a subset of contracts $X' \subseteq X$, again equivalently defined for member states and wait times. An **allocation** $Y \subseteq X$ is a set of contracts with $|Y_a| \leq 1$ for all $a \in A$, and $|Y_m \cap Y_w| \leq r_m^w$ for all $m, w \in M \times W$. Let $\mathscr{Y}$ denote the set of all allocations. Slightly abusing notation, we will use P_a for preferences over contracts and allocations.[7]

3 Member State Choice Rule

A choice rule for a member state makes a selection out of a set of contracts based on the member state's priority ordering, burden-sharing quota, and bureaucratic capacities. A **choice rule** $C_m : \mathscr{X} \mapsto \mathscr{X}$ associates with each subset of contracts $X' \in \mathscr{X}$ a subset of contracts $C_m(X') \in \mathscr{X}$ such that $C_m(X') \subseteq X'_m$.

We say that, an asylum seeker $a \in A(X')$ **qualifies for acceptance** under $X' \in \mathscr{X}$ if

$$\sum_{\{a' \in A(C_m(X')) : a' \pi_m a\}} s(a') < q_m.$$

For a given choice rule and a subset of contracts, consider the set of asylum seekers with at

[5] P_a is a strict simple order, that is, a binary related that is *transitive*, *asymmetric*, and *complete* (ranks everything except (x, x) — a contract with itself). The associated simple order R_a is *transitive*, *antisymmetric*, and *strongly complete* (ranks everything).

[6] One might assume that asylum seekers prefer lower wait times for the same member state. However, we allow for more general preferences to accommodate the concept of humanitarian visas. This means asylum seekers could opt for longer wait times outside the EU to allow enough time to apply for a humanitarian visa, which requires a designated member state at the time of application, and then travel there after the centralized match.

[7] For any $x, x' \in X$ we say $x \; P_a \; x' \iff (m_x, w_x) \; P_a \; (m_{x'}, w_{x'})$, and for any $Y, Y' \in \mathscr{Y}$ we say $Y \; P_a \; Y' \iff Y_a \; P_a \; Y'_a$.

least one contract accepted. An asylum seeker qualifies for acceptance if the total burden-size of already accepted asylum seekers with a higher priority is strictly less than the member state's burden-sharing quota.

We say that, an asylum seeker $a \in A(X')$ **qualifies for wait time** $w \in W$ under $X' \in \mathscr{X}$ if $X'_a \cap X'_w \neq \varnothing$ and

$$|\{a' \in A(C_m(X')_w) : a'\pi_m a\}| < r_m^w.$$

For a given choice rule, wait time, and a subset of contracts, consider the set of asylum seekers with a contract accepted for the relevant wait time. An asylum seeker qualifies for that wait time if the number of higher priority asylum seekers scheduled for that wait time is strictly less than the relevant bureaucratic capacity.

With this in mind, we propose a choice rule for member states that satisfies the following three properties:

- A choice rule C_m satisfies **feasibility** if for all $X' \in \mathscr{X}$, for all $a \in A(X')$, and for all $w \in W$, we have

  $$|C_m(X')_a| \leq 1 \text{ and } |C_m(X')_w| \leq r_m^w.$$

  A choice rule is feasible if at most one contract is accepted per asylum seeker and the member state's bureaucratic capacities are not violated. This is unquestionably essential and is often assumed.

- A choice rule C_m satisfies **early filling** if for all $x \in C_m(X')$, and $x' \notin C_m(X')$, such that $a_x = a_{x'} = a$ and $w_{x'} < w_x$, the asylum seeker does not qualify for the lower wait time $w_{x'}$.

  For a given subset of contracts, a choice rule satisfies early filling if for any accepted contract there does not exists a lower wait time contract for which there is still bureaucratic capacity left. This minimizes overall wait times at the member state.

- A choice rule C_m **respects member state priorities** if for all $X' \in \mathscr{X}$ an asylum seeker $a \in A(X')$ ends up with a contract if and only if she qualifies for acceptance and a wait time.

  This property ensures that if an asylum envies another asylums seekers' contracts, then that asylum seeker must have a higher priority.

Our main result in this section shows that there is a unique choice rule that satisfies feasibility, early filling, and respects member state priorities. The choice rule is straightforward and intuitive: asylum seekers are processed one by one in order of priority. Each is assigned the contract with the shortest waiting time that still has sufficient bureaucratic capacity. The process continues until either all asylum seekers have been matched or the burden-sharing quota is fully met, at which point no further contracts are accepted.

Member state choice rule $\hat{C}_m$ for a set of contract $X' \subseteq X$:
Step $k \geq 1$:

1. Let X^{k-1} denote the set of accepted contracts with $X^0 = \varnothing$, and let $Z^{k-1} = \{x \in X'_m : r_m^{w_x} > |X_{w_x}^{k-1}| \text{ and } a_x \notin A(X^{k-1})\}$ denote the set of contracts specifying an asylum seeker not yet accepted with a wait time that has bureaucratic capacity left.
   If either the burden-sharing quota is reached $\sum_{x \in X^{k-1}} s(a_x) \geq q_m$ or no acceptable contract is left $Z^{k-1} = \varnothing$ the algorithm ends and $\hat{C}_m(X') = X^{k-1}$.

2. Otherwise, determine the highest priority asylum seeker left a^k, defined as $a \in A(Z^{k-1})$ such that $a \; \pi \; a'$ for all $a' \in A(Z^{k-1}) \setminus \{a\}$.
   Then, accept the lowest wait time contract x^k available for that asylum seeker, defined as $x \in Z_{a^k}^{k-1}$ such that $w_x < w_{x'}$ for all $x' \in Z_{a^k}^{k-1} \setminus \{x\}$.
   Adjust the set of accepted contracts $X^k = X^{k-1} \cup \{x^k\}$ and proceed to step $k+1$.

Hereafter, we will use $\hat{C}_m$ as the relevant choice rule, which is characterized in following result. Proof of Theorem 1 is relegated to Appendix A.

Theorem 1. *A choice rule C_m satisfies feasibility, early filling, and respects member state priorities if and only if it is $\hat{C}_m$.*

4 A Stable and Strategy-Proof Mechanism

In this section, we introduce a mechanism with desirable properties designed to assign asylum seekers to member states and manage wait times. We will start with the relevant definitions:

- A **mechanism** is a function φ that assigns every asylum seeker matching problem $\langle s, P \rangle$ an allocation $\varphi(s, P) \in \mathscr{Y}$.

- A mechanism φ is (pairwise) **stable** if for problem $\langle s, P \rangle$ if for every pair $a \in A$, $m \in M$ and contract $x \in X \setminus \varphi(s, P)$,

$$x \; P_a \; \varphi(s, P) \implies x \notin C_{m_x}(\varphi(s, P) \cup \{x\}).$$

  A mechanism is considered (pairwise) stable if, whenever an asylum seeker prefers a contract over the one assigned to them by the mechanism, the member state involved would reject this alternative contract when evaluated alongside its existing set of assigned contracts. Thus, stability ensures fairness by preventing any party from benefiting at the expense of another.

- A mechanism φ is **strategy-proof** (for asylum seekers) for problem $\langle s, P\rangle$, if for all $a \in A$, $\hat{P}_a \in \mathscr{P}_a$, we have

$$\varphi(s, P)\ R_a\ \varphi(s, \hat{P}_a, P_{-a}).$$

  A mechanism is strategy-proof if for every asylum seeker, revealing her true preferences is a weakly dominant strategy.

4.1 Homogenous Burden-size

If asylum seekers are treated identically in terms of burden-sizes, then a stable and strategy-proof mechanism exists, namely, the asylum-seeker-proposing cumulative offer mechanism. We start by defining this mechanism:

Asylum seeker proposing cumulative offer mechanism φ^c:

Step $k \geq 1$.

1. Let X^k denote the set of proposed contacts with $X^0 = \varnothing$.
   A contract $x \in X^{k-1}$ is tentatively accepted if $x \in C_{m_x}(X^{k-1})$ and rejected otherwise. If there is an asylum seeker with no contract tentatively accepted, let some asylum seeker a propose her most preferred contract, following P_a, among contracts that have not been proposed $x^k \in X \setminus X^k$. Then, set $X^k = X^{k-1} \cup \{x^k\}$.

2. Otherwise, the process terminates with $\varphi^c(s, P) = \bigcup_{m \in M} C_m(X^{k-1}_m)$.

Theorem 2. *Suppose every member state is equipped with choice rule* $\hat{C}_m$*. Then the asylum-seeker-proposing cumulative offer mechanism is stable and strategy-proof for any problem* $\langle s, P\rangle$ *with identical burden-sizes across asylum seekers.*

A centralized system can assign asylum seekers to member states in a stable and (asylum-seeker) strategy-proof manner, provided families and individuals are treated equally in the burden-sharing quota during the assignment process. This requires identical treatment of applications from individuals and families. While the additional burden of supporting families cannot be factored in without compromising stability and strategy-proofness, imbalances can be addressed over time. Since asylum seeker matching is a recurring event, member states accepting more families could have their future burden-sharing commitments adjusted downward, balancing the overall burden.

Proving Theorem 2 requires conditions on the member state choice rule, which will together ensure that the above defined cumulative offer mechanism is stable and strategy-proof.

- A choice rule C_m satisfies **substitutablility** if for all $X' \subseteq X$, and $x, x' \in X \setminus X'$, we have

$$x \in C_m(X' \cup \{x, x'\}) \implies x \in C_m(X' \cup \{x\}).$$

A choice rule is substitutable if whenever a contract is chosen from a set of contracts, then the contract is also selected from any subset of contracts containing that contract.

- A choice rule C_m satisfies **irrelevance of rejected contracts** if for all $X' \subseteq X$, and $x \in X \setminus X'$,

$$x \notin C_m(X' \cup \{x\}) \implies C_m(X' \cup \{x\}) = C_m(X').$$

A choice rule satisfies irrelevance of rejected contracts if removing a contract that has not been chosen does not affect the set of chosen contracts.

- A choice rule C_m satisfies the **law of aggregate demand** if for all $X' \subseteq X$, and $x \in X \setminus X'$ we have

$$|C_m(X' \cup \{x\})| \geq |C_m(X')|.$$

A choice rule satisfies the law of aggregate demand if the set of chosen contracts weakly increases with the set of available contracts.

Hatfield and Milgrom (2005) and Aygün and Sönmez (2013) show that substitutability and irrelevance of rejected contracts are sufficient conditions on the choice rule for the cumulative offer mechanism to be stable. Additionally, combining these conditions with the law of aggregate demand ensures strategy-proofness of the cumulative offer mechanism.

Proposition 1. *The choice rule $\hat{C}_m$ satisfies the irrelevance of rejected contracts condition.*

However, wait times introduce complementarities regardless of the specified burden-sizes for asylum seekers. Therefore, substitutability and law of aggregate demand are violated for the choice rule $\hat{C}_m$ even if the asylum seeker matching problem $\langle s, P\rangle$ specifies the same burden-size for every asylum seeker. We present Example 1 to illustrate this point.

Example 1 (Choice rule violates substitutability and the law of aggregate demand)**.** Consider $A = \{a_1, a_2\}$, $M = \{m\}$, $W = \{w_l, w_h\}$. Let $s(a_1) = s(a_2) = 1$, $q_m = 2$, $r_m^{w_l} = r_m^{w_l} = 1$,

$x_1 = (a_1, m, w_l)$, $x_2 = (a_1, m, w_h)$,

$x_3 = (a_2, m, w_l)$, $x_4 = (a_2, m, w_h)$, and

$\pi_m : a_1 - a_2$.

To see that $\hat{C}_m$ violates substitutability, take $X' = \{x_2\}$, x_4, and x_1. We have that $x_4 \in \hat{C}_m(X' \cup \{x_4, x_1\}) = \{x_1, x_4\}$ while $x_2 \notin \hat{C}_m(X' \cup \{x_4\}) = \{x_2\}$.Similarly, we can find a violation of the law of aggregate demand for $X' = \{x_2, x_3\}$, and x_1. We have $|\hat{C}_m(X')| = |\{x_2, x_3\}| > |\hat{C}_m(X' \cup \{x_1\})| = |\{x_1\}|$.

Remark 1. *The choice rule in Example 1 also violates the weaker unilateral substitutes condition (Hatfield and Kojima, 2010), which together with the law of aggregate demand is sufficient for stability and strategy-proofness (for asylum seekers). A choice rule C_m is unilateral substitutable if there does not exist $X' \subseteq X$ and $x, x' \in X \setminus X'$ such that $a_x \notin A(X')$, and if $x \in C_m(X' \cup \{x, x'\})$ then $x \in C_m(X' \cup \{x\})$. We have a violation of unilateral substitutability as $a_{x_4} = a_2 \notin A(X') = \{a_1\}$.*

The literature has discussed several ways to relax the substitutability condition, as well as cases in which violations of substitutability and the law of aggregate demand are not harmful for stability and strategy-proofness (Hatfield and Kojima, 2010; Kominers and Sönmez, 2016; Hatfield and Kominers, 2019; Hatfield et al., 2020). In this paper, we rely on the result of Hatfield and Kominers (2019) which requires us to define a new property:

- C'_m is a **completion** of a choice rule C_m, if for all $X' \subseteq X$, either $C'_m(X') = C_m(X')$, or there exist distinct contracts $x, x' \in C'_m(X')$ that are associated with the same asylum seeker, that is, $a(x) = a(x')$.

  A choice rule completes another choice rule if they both choose the same set of contracts whenever the selected set of the completing choice rule is feasible.

Hatfield and Kominers (2019) show that if completion of a choice rule satisfies substitutability and the law of aggregate demand (together with the irrelevance of rejected contracts) then the asylum-seeker-proposing cumulative offer mechanism is stable and strategy-proof (for asylum seekers). The result holds even if the cumulative offer mechanism uses the original choice rule, which violates both properties.

Theorem. *(Hatfield and Kominers, 2019) If, for each $m \in M$, the choice function C_m has a substitutable completion C'_m that satisfies the law of aggregate demand and the irrelevance of rejected contracts condition, then the cumulative offer mechanism is stable and strategy-proof (for asylum seekers).*

The following two restrictions on priority orderings will be useful before proceeding:

- π_m satisfies **small burden-size priority** if for all $a, a' \in A$, we have

$$a \pi_m a' \implies s(a) \leq s(a').$$

If an asylum seeker has higher priority than another asylum seeker, then she must have a weakly smaller burden-size.

- π_m satisfies **large burden-size priority** if for all $a, a' \in A$, we have

$$a \pi_m a' \implies s(a) \geq s(a').$$

If an asylum seeker has higher priority than another asylum seeker, then she must have a weakly larger burden-size.

We next describe a choice rule that is a completion of the choice rule characterized in Theorem 1.

Completion of the member state choice rule $\hat{C}'_m$:
Step $k \geq 1$:

1. Let X^k denote the set of accepted contracts, with $X^0 = \varnothing$.
   Similarly, let $Z^k = \{x \in X'_m : r_m^{w_x} > |X^k_{w_x}| \text{ and } x \notin X^k\}$ denote the set of still acceptable contracts.
   If either the burden-sharing quota is reached $\sum_{x \in X^{k-1}} s(a_x) \geq q_m$, or no acceptable contract is left $Z^{k-1} = \varnothing$ the algorithm ends and $\hat{C}_m(X') = X^{k-1}$.

2. Otherwise, determine the highest priority asylum seeker left a^k, defined as $a \in A(Z^{k-1})$ such that $a \;\pi\; a'$ for all $a' \in A(Z^{k-1}) \setminus \{a\}$.
   Then, accept the lowest wait time contract x^k available for that asylum seeker, defined as $x \in Z^{k-1}_{a^k}$, such that $w_x < w_{x'}$ for all $x' \in Z^k_{a^k} \setminus \{x\}$.
   Adjust the set of accepted contracts $X^k = X^{k-1} \cup \{x^k\}$ and proceed to step $k+1$.

The only difference of the completion $\hat{C}'_m$ relative to the original choice rule $\hat{C}_m$ is that an asylum seeker already holding a contract still remains in the race for more contracts. In particular, all contracts of a higher priority asylum seeker are accepted before all contracts with the identical wait time of a lower priority asylum seeker (Lemma 1).

Lemma 1. *$\hat{C}'_m$ is a completion of $\hat{C}_m$.*

The completion $\hat{C}'_m$ satisfies the irrelevance of rejected contracts (Proposition 2), is substitutable if the member states' priority order satisfies large burden-size priority (Proposition 3), and satisfies the law of aggregate demand if small burden-size priority holds (Proposition 4).

Proposition 2. *The completion $\hat{C}'_m$ satisfies irrelevance of rejected contracts.*

Proposition 3. *The completion $\hat{C}'_m$ satisfies substitutability if π_m satisfies large-size priority.*

Proposition 4. *The completion $\hat{C}'_m$ satisfies the law of aggregate demand if π_m satisfies small burden-size priority.*

When the burden-size is identical across asylum seekers, the large and small burden-size conditions hold simultaneously. Therefore, these results collectively demonstrate that the cumulative offer mechanism is stable and strategy-proof (Theorem 2). Proofs of Propositions 1-4 and Lemma 1 are provided in Appendix A.

We revisit Example 1 to show that the completion addresses issues related to substitutability and the law of aggregate demand caused by bureaucratic capacities, provided the burden-size is homogeneous (Example 2).

Example 2 (Completion Example 1 revisited)**.** Recall the set-up from Example 1: $A = \{a_1, a_2\}$, $M = \{m\}$ $W = \{w_l, w_h\}$, $s(a_1) = s(a_2) = 1$, $q_m = 2$, $r_m^{w_l} = r_m^{w_l} = 1$, and

$$x_1 = (a_1, m_1, w_l),\ x_2 = (a_1, m_1, w_h),$$

$$x_3 = (a_2, m_1, w_l),\ x_4 = (a_2, m_1, w_h),$$

$$\pi_m : a_1 - a_2.$$

Revisiting the violation of substitutability for $X' = \{x_2\}$, x_4, and x_1. We have that $\hat{C}'_m(X' \cup \{x_4, x_1\}) = \{x_1, x_2\}$ as well as $\hat{C}'_m(X' \cup \{x_4\}) = \{x_2\}$. Similarly, revisiting the law of aggregate demand violation for $X' = \{x_2, x_3\}$ and x_1, we have $|\hat{C}'_m(X')| = |\{x_2, x_3\}| = |\hat{C}'_m(X' \cup \{x_1\})| = |\{x_1, x_2\}|$.

4.2 Heterogenous Burden-size

In the following two examples we illustrate the problem in using the cumulative offer mechanism when the burden-size is not identical across asylum seekers. If the member states' priority order does not satisfy large burden-size priority, Example 3 shows that there is no guarantee for the existence of a completion satisfying substitutability. If the member states' priority order does not satisfy small burden-size priority, Example 4 shows that there is no guarantee for the existence of a completion satisfying the law of aggregate demand.

Example 3 (No completion satisfying substitutability)**.** Consider $A = \{a_1, a_2, a_3\}$, $M = \{m\}$, $W = \{w_l, w_h\}$. Let $s(a_1) = 1$, $s(a_2) = s(a_3) = 2$, $q_m = 2$, $r_m^{w_l} = r_m^{w_l} = 1$,

$$x_1 = (a_1, m, w_l),$$

$$x_2 = (a_2, m, w_l),$$

$x_3 = (a_3, m, w_h)$, and

$\pi_{m_1} : a_1 - a_2 - a_3$.

Note that priorities violate large burden-size priority. Moreover, for any $X' \subseteq X$, such that $|X'_a| \leq 1$ for all $a \in A$, any completion $\hat{C}'_m$ of $\hat{C}_m$ must choose the same set of contracts as $\hat{C}_m$, that is, $\hat{C}'_m(X') = \hat{C}_m(X')$. We have that any completion must choose $\hat{C}'_m(\{x_2, x_3\}) = \{x_2\}$ and $\hat{C}'_m(\{x_1, x_2, x_3\}) = \{x_1, x_3\}$ which violates substitutability.

Remark 2. *With Example 3's setup we can construct a counterexample for all sufficient conditions ensuring the existence of a stable mechanism. We have that $\hat{C}_m(\{x_2, x_3\}) = \{x_2\}$ and $\hat{C}_m(\{x_1, x_2, x_3\}) = \{x_1, x_3\}$, a violation of bilateral substitutability and therefore also of unilateral substitutability as well as substitutability (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2010). Note that a_3 gets a w_h-slot if $\{x_1, x_2, x_3\}$ are proposed but foregoes the slot if only $\{x_2, x_3\}$ are proposed, so we cannot construct an associated one-to-one market as in Kominers and Sönmez (2016). Finally, x_2, x_3, x_1 is an observable offer process, and thus we have a violation of observable substitutability (Hatfield et al., 2020), because $\hat{C}_m(\{x_2, x_3\}) = \{x_2\}$ but $\hat{C}_m(\{x_1, x_2, x_3\}) = \{x_1, x_3\}$.*

Example 4 (No completion satisfying the law of aggregate demand)**.** Consider $A = \{a_1, a_2, a_3\}$, $M = \{m\}$, $W = \{w_l\}$. Let $s(a_1) = 2$, $s(a_2) = s(a_3) = 1$, $q_m = 2$, $r_m^{w_l} = 2$, and

$x_1 = (a_1, m, w_l)$,

$x_2 = (a_2, m, w_l)$,

$x_3 = (a_3, m, w_l)$,

$\pi_m : a_1 - a_2 - a_3$.

Note that priorities violate small burden-size priority. Moreover, for any $X' \subseteq X$, such that $|X'_a| \leq 1$ for all $a \in A$, any completion $\hat{C}'_m$ of $\hat{C}_m$ must choose the same set of contracts as $\hat{C}_m$, that is, $\hat{C}'_m(X') = \hat{C}_m(X')$. It follows that, any completion must choose $\hat{C}'_m(\{x_2, x_3\}) = \{x_2, x_3\}$ and $\hat{C}'_m(\{x_1, x_2, x_3\}) = \{x_1\}$ which violates the law of aggregate demand.

Remark 3. *Note that x_2, x_3, x_1 is an observable offer process. Thus Example 4 also shows a violation of observable size monotonicity (Hatfield et al., 2020). The same is true for Example 3, which shows a violation of observable substitutability. Thus the weakest known conditions for existence of a stable and strategy-proof mechanism are violated.*

In general, when the burden-size varies across asylum seekers, there does not exist a stable mechanism (Example 5), nor can we ensure strategy-proofness to hold for a mechanism that selects a stable outcome, whenever one exists (Example 6).

Theorem 3. *For a problem $\langle s, P \rangle$ with heterogenous burden-sizes across asylum seekers, a stable and strategy-proof mechanism may not exist.*

We present two examples to prove Theorem 3. We will write $W = \{w_l, w_h\}$ when we mean that $w_l < w_h$, where l indicates low wait time and h indicates high wait time.

Example 5 (No stable outcome)**.** Consider $A = \{a_1, a_2, a_3\}$, $M = \{m_1, m_2, m_3, m_4\}$, and $W = \{w_l, w_h\}$. Let $s(a_1) = 1$, $s(a_2) = s(a_3) = 2$, $q_{m_1} = 2$, $q_{m_2} = q_{m_3} = q_{m_4} = 1$, $r_m^{w_h} = r_m^{w_l} = 1$ for all $m \in M$,

$$P_{a_1} : x_1 = (a_1, m_2, w_l) - x_2 = (a_1, m_1, w_l) - \ldots,$$

$$P_{a_2} : x_3 = (a_2, m_1, w_l) - x_4 = (a_2, m_3, w_l) - \ldots,$$

$$P_{a_3} : x_5 = (a_3, m_1, w_h) - x_6 = (a_3, m_2, w_h) - \ldots,$$

$$\pi_{m_1} : a_1 - a_2 - a_3,$$

$$\pi_{m_2} : a_3 - a_2 - a_1,$$

$$\pi_{m_3} : a_1 - a_2 - a_3, \text{ and}$$

$$\pi_{m_4} : \ldots.$$

m_4 makes sure that the constraint $\sum_{m \in M} q_m \geq \sum_{a \in A} s(a)$ is satisfied. Since asylum seeker a_1 has the highest priority in member state $m_1 = m_{x_2}$, in any stable allocation a_1 must end up with either x_1 or x_2. Suppose a_1 ends up with her top choice x_1 at member state $m_2 = m_{x_1}$. In this case a_2 must get her top choice x_3, since she has the highest priority among the remaining asylum seekers at member state $m_1 = m_{x_3}$, preventing a_3 from getting x_5 with member state $m_1 = m_{x_5}$. In turn, a_3 forms a blocking pair with member state $m_2 = m_{x_6}$ through contract x_6. Suppose a_1 gets x_2 instead. Then a_2 can no longer get her top choice x_3, due to insufficient bureaucratic capacity $r_m^{w_l} = 1$ of member state $m_1 = m_{x_3}$. On the other hand a_3 can get her first choice x_5 as there is sufficient bureaucratic capacity for the high wait time $r_m^{w_h} = 1$ of member state $m_1 = m_{x_5}$, while a_2 gets x_4 her second choice. But now m_2 has no asylum seeker assigned and a_1 forms a blocking pair with member state $m_2 = m_{x_1}$ through contract x_1. Thus, the non-existence of stable outcomes arises from a combination of burden-sizes and bureaucratic capacities.

Example 6 (No strategy-proof outcome)**.** Consider $A = \{a_1, a_2, a_3\}$, $M = \{m_1, m_2, m_3\}$, and $W = \{w_l\}$. Let $s(a_1) = 2$, $s(a_2) = s(a_3) = 1$, $q_{m_1} = 2$, $q_{m_2} = q_{m_3} = 1$, $r_m^{w_l} = 2$ for all $m \in M$, and

$$P_{a_1} : x_1 = (a_1, m_2, w_l) - x_2 = (a_1, m_1, w_l) - \dots,$$

$$P_{a_2} : x_3 = (a_2, m_2, w_l) - x_4 = (a_2, m_1, w_l) - x_5 = (a_2, m_3, w_l),$$

$$P_{a_3} : x_6 = (a_3, m_1, w_l) - x_7 = (a_3, m_2, w_l) - \dots,$$

$$\pi_{m_1} : a_1 - a_2 - a_3,$$

$$\pi_{m_2} : a_3 - a_2 - a_1,$$

$$\pi_{m_3} : a_1 - a_2 - a_3.$$

In this example, a problem arises because either both a_2 and a_3 are matched to m_1, in which case a_1 gets her top choice, or a_1 is matched to m_1 by herself. Because a_2 blocks the former with x_3, the latter is the unique stable allocation $Y_1 = \{x_2, x_5, x_7\}$. If a_2 changes her preference to $\hat{P}_{a_2} : x_4 - x_5 - x_3$, the former allocation is no longer blocked — in fact, $Y_2 = \{x_1, x_4, x_6\}$ becomes the unique stable allocation. It follows that even if there exists a mechanism φ that selects a stable outcome whenever one exists, strategy-proofness is violated for asylum seekers as $\varphi(s, P_{a_1}, \hat{P}_{a_2}, P_{a_3}) = Y_2 \; P_{a_2} \; \varphi(s, P) = Y_1$.

5 Conclusion

We presented an alternative to the current decentralized asylum assignment to effectively match asylum seekers to member states, taking into account the complex interplay of preferences, priorities, capacities, and burden-sharing commitments. We demonstrate that a centralized mechanism can ensure stability and strategy-proofness under conditions of equal burden-sizes. However, when the burden-sizes differ, the feasibility of achieving these desirable properties is challenged.

While the homogeneous burden-size assumption simplifies the problem and leads to a desirable solution, it might not reflect the real-world complexities of asylum assignment, where some member states may face disproportionately higher numbers of asylum seekers with families. To address the challenges posed by heterogeneous burden-sizes, one may consider adjusting burden-sharing quotas over time by rewarding member states that accept a disproportionate number of families by lowering their quotas in subsequent assignment rounds. Such an approach could help mitigate the long-term effects of burden imbalances. Another possibility worth exploring could be designing mechanisms that relax either stability or strategy-proofness to accommodate heterogenous burden-sizes. Previous studies, such as Nguyen and Vohra (2018) and Delacrétaz (2019), have already hinted at the potential trade-offs and implications of such relaxations, offering avenues for future research.

In terms of practicality, our proposed mechanism can be implemented at regular intervals as a recurring process, allowing the batch of newly arrived asylum seekers to incorporate their preferences into the asylum assignment process. This approach is similar to Japan's centralized daycare center allocation, which is not based on a first-come, first-served basis. Instead, slots are assigned annually in April via a centralized algorithm that considers family preferences and prioritizes children from low-income households, single-parent families, or those with guardians facing health challenges (Sun et al., 2023). German municipalities also allocate daycare places at regular intervals.[8] Another example of this type of matching mechanism is Singapore's Build-To-Order (BTO) system for public housing, where flats are allocated monthly based on buyers' preferences and a priority system. While asylum seekers arrive continuously, processing applications in batches at regular intervals could significantly improve the current decentralized system.

[8] See https://kitamatch.com/.

References

Abdulkadiroğlu, Atila and Tayfun Sönmez, "School choice: A mechanism design approach," *American Economic Review*, 2003, *93* (3), 729–747.

Ahani, Narges, Paul Gölz, Ariel D. Procaccia, Alexander Teytelboym, and Andrew C. Trapp, "Dynamic Placement in Refugee Resettlement," *Operations Research*, 2024, *72* (3), 1087–1104.

_ , Tommy Andersson, Alessandro Martinello, Alexander Teytelboym, and Andrew C Trapp, "Placement optimization in refugee resettlement," *Operations Research*, 2021, *69* (5), 1468–1486.

Andersson, Tommy and Lars Ehlers, "Assigning refugees to landlords in Sweden: Efficient, stable, and maximum matchings," *The Scandinavian Journal of Economics*, 2020, *122* (3), 937–965.

_ , _ , and Alessandro Martinello, "Dynamic refugee matching," 2018.

Aygün, Orhan and Bertan Turhan, "Dynamic reserves in matching markets," *Journal of Economic Theory*, 2020, *188*, 105069.

_ and Tayfun Sönmez, "Matching with contracts: Comment," *American Economic Review*, 2013, *103* (5), 2050–2051.

Balinski, Michel and Tayfun Sönmez, "A tale of two mechanisms: student placement," *Journal of Economic Theory*, 1999, *84* (1), 73–94.

BBC, "UK to accept 20,000 refugees from Syria by 2020," 2015.

Beck, Gunnar, Nuale Mole, and Marcelle Reneman, "The application of the EU Charter of Fundamental Rights to asylum procedure law, European Council on Refugees and Exiles," 2014.

Biró, Péter and Eric McDermid, "Matching with sizes (or scheduling with processing set restrictions)," *Discrete Applied Mathematics*, 2014, *164*, 61–67.

Caspari, Gian, "An alternative approach to asylum assignment," 2019.

Dean, Brian C, Michel X Goemans, and Nicole Immorlica, "The unsplittable stable marriage problem," in "Fourth IFIP International Conference on Theoretical Computer Science-TCS 2006: IFIP 19th Worm Computer Congress, TC-1, Foundations of Computer Science, August 23–24, 2006, Santiago, Chile" Springer 2006, pp. 65–75.

Delacrétaz, David, "Stability in matching markets with sizes," 2019.

— , **Scott Duke Kominers, and Alexander Teytelboym**, "Matching mechanisms for refugee resettlement," *American Economic Review*, 2023, *113* (10), 2689–2717.

Dimakopoulos, Philipp D and C-Philipp Heller, "Matching with waiting times: The German entry-level labor market for lawyers," *Games and Economic Behavior*, 2019, *115*, 289–313.

European Commission, "Proposal for establishing the criteria and mechanisms for determining the Member State responsible for examining an application for international protection lodged in one of the Member States by a third-country national or a stateless person," Technical Report 2016.

European Council of Human Rights, "European Convention on Human Rights," 2013.

Hassidim, Avinatan, Assaf Romm, and Ran I Shorrer, "Redesigning the Israeli psychology master's match," 2017, pp. 121–122.

Hatfield, John William and Fuhito Kojima, "Substitutes and stability for matching with contracts," *Journal of Economic Theory*, 2010, *145* (5), 1704–1723.

— **and Paul R Milgrom**, "Matching with contracts," *American Economic Review*, 2005, *95* (4), 913–935.

— **and Scott Duke Kominers**, "Hidden substitutes," 2019.

— , — , **and Alexander Westkamp**, "Stability, Strategy-Proofness, and Cumulative Offer Mechanisms," *The Review of Economic Studies*, 09 2020, *88* (3), 1457–1502.

Jones, Will and Alexander Teytelboym, "Choices, preferences and priorities in a matching system for refugees," *Forced Migration Review*, 2016, (51).

— **and** — , "The international refugee match: A system that respects refugees' preferences and the priorities of states," *Refugee Survey Quarterly*, 2017, *36* (2), 84–109.

— **and** — , "Matching systems for refugees," *Journal on Migration and Human Security*, 2017, *5* (3), 667–681.

— **and** — , "The local refugee match: Aligning refugees' preferences with the capacities and priorities of localities," *Journal of Refugee Studies*, 2018, *31* (2), 152–178.

Jr, Alexander S Kelso and Vincent P Crawford, "Job matching, coalition formation, and gross substitutes," *Econometrica*, 1982, pp. 1483–1504.

Kominers, Scott Duke and Tayfun Sönmez, "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 2016, *11* (2), 683–710.

Maiani, Francesco, "The Reform of the Dublin III Regulation," Technical Report, Parlement européen 2016.

McDermid, Eric J and David F Manlove, "Keeping partners together: algorithmic results for the hospitals/residents problem with couples," *Journal of Combinatorial Optimization*, 2010, *19*, 279–303.

Moraga, Jesús Fernández-Huertas and Hillel Rapoport, "Tradable immigration quotas," *Journal of Public Economics*, 2014, *115*, 94–108.

Nguyen, Thanh and Rakesh Vohra, "Near-feasible stable matchings with couples," *American Economic Review*, 2018, *108* (11), 3154–3169.

Schuck, Peter H, "Refugee Burden-Sharing: A Modest Proposal," *Yale Journal of International Law*, 1997, *22.*

Sönmez, Tayfun and Tobias B Switzer, "Matching with (branch-of-choice) contracts at the United States Military Academy," *Econometrica*, 2013, *81* (2), 451–488.

Sun, Zhaohong, Yoshihiro Takenami, Daisuke Moriwaki, Yoji Tomita, and Makoto Yokoo, "Daycare matching in Japan: Transfers and siblings," in "Proceedings of the AAAI Conference on Artificial Intelligence," Vol. 37 2023, pp. 14487–14495.

Yenmez, M Bumin, "A college admissions clearinghouse," *Journal of Economic Theory*, 2018, *176*, 859–885.

A Mathematical Appendix and Proofs

Theorem 1

Proof. We start with the **if** direction.

Feasibility: Suppose feasibility is violated and consider the first step k in which some asylum seeker a^k gets her second contract. That is, $a^k \in A(Z^{k-1})$ such that $a^k \pi a'$ for all $a' \in A(Z^{k-1}) \setminus \{a^k\}$. Note that $Z^{k-1} = \{x \in X'_m : r_m^{w_x} > |X_{w_x}^{k-1}|$ and $a_x \notin A(X^{k-1})\}$ but by assumption we have $a^k \in A(X^{k-1})$ as one contract has already been accepted: leading to a contradiction. Similarly, consider the first step k, in which $|X_w^{k-1}| = r_m^w$ and $w_{x^k} = w$, again we get a contradiction with $a^k \in A(Z^{k-1})$ as $r_m^{w_x} = |X_{w_x}^{k-1}|$.

Early filling: Consider some asylum seeker a^k who got assigned contract x^k at step k. Note that the algorithm chooses the lowest contract in $Z_{a^k}^{k-1}$, hence for $x' \in X_{a^k}$ with $w_{x'} < w_{x^k}$ we have that $r_m^{w_{x'}} = |X_{w'_x}^{k-1}|$. Since all asylum seekers in X^{k-1} have higher priority than x^k, it follows directly that $|\{a' \in A(X_{w'_x}^{k-1}) : a' \pi a\}| = |\{a' \in A(C_m(X')_{w_{x'}}) : a' \pi a\}| = r_m^{w_x}$, and thus early filling is satisfied.

Respecting member state priorities: Consider a^k accepted at step k. We have that $\sum_{x \in X^{k-1}} s(a_x) < q_m$ and since a^k has higher priority than all remaining asylum seekers, we have $\sum_{x \in X^{k-1}} s(a_x) = \sum_{\{a' \in A(C_m(X')) : a' \pi_m a\}} s(a') < q_m$.

Similarly, if the algorithm ends at step k, we have that $\sum_{x \in X^{k-1}} s(a_x) \geq q_m$ and as a^k has lower priority than all previously chosen asylum seekers, we have $\sum_{x \in X^{k-1}} s(a_x) = \sum_{\{a' \in A(C_m(X')) : a' \pi_m a\}} s(a') \geq q_m$.

In step k, only contracts $Z^{k-1} = \{x \in X'_m : r_m^{w_x} > |X_{w_x}^k|$ and $a_x \notin A(X^k)\}$ are considered. Take any unassigned asylum seekers $a \in A(X') \setminus \{a^1, \ldots, a^{k-1}\}$: clearly any contract $x \in X'_a \setminus Z^{k-1}$ does not qualify for a wait time as the bureaucratic capacity is already occupied with higher priority asylum seekers, $|X_{w_x}^{k-1}| = |\{a' \in A(X_{w_x}^{k-1}) : a' \pi a\}| = |\{a' \in A(C_m(X')_{w_x}) : a' \pi a\}| = r_m^{w_x}$. Similarly, if a^k is accepted in step k then $x^k \in Z^{k-1}$ and hence $|X_{w_x}^{k-1}| = |\{a' \in A(X_{w_x}^{k-1}) : a' \pi a^k\}| = |\{a' \in A(C_m(X')_{w_x}) : a' \pi a^k\}| \leq r_m^{w_x}$.

Hence, almost by construction, an asylum seeker is accepted if and only if she qualifies for acceptance and a wait time.

We show the **only if** direction, proceeding by induction. Suppose the described algorithm for determining $\hat{C}_m(X')$ stops after k steps, and therefore $\hat{C}_m(X') = X^{k-1}$.

Base step. If the algorithm stops at step 1 then $\hat{C}_m(X') = C_m(X') = X^0$; otherwise, $X^1 \subseteq C_m(X')$.

Suppose the algorithm stops at step 1.

We have $\sum_{x \in X^0} s(a_x) = \sum_{x \in \varnothing} s(a_x) = 0 \geq q_m$. It follows that no asylum seeker *qualifies for acceptance*, since for all $a \in A(X')$ we have $\sum_{\{a' \in A(C_m(X')) : a' \pi_m a\}} s(a') \geq q_m$. It follows that

$\hat{C}_m(X') = C_m(X') = X^0 = \varnothing$ since $C_m(X')$ satisfies *respecting member state priorities*.

Suppose the algorithm does not stop at step 1.

No contract in $X' \setminus Z^0$ can ever be accepted without violating *feasibility*, hence the only relevant asylum seekers are in the set $A(Z^0)$. Moreover, as the algorithm did not stop, we have $\sum_{x \in X^0} s(a_x) = \sum_{x \in \varnothing} s(a_x) = 0 < q_m$ and $Z^0 \neq \varnothing$. As a^1 is the highest priority asylum seeker, we have that $\sum_{\{a' \in A(C_m(X')): a' \pi_m a\}} s(a') < q_m$ and hence a^1 *qualifies for acceptance*. Similarly, $|\{a' \in A(C_m(X')_{w_x}) : a' \pi a^1\}| \leq r_m^{w_x}$ holds for all $x \in Z^0_{a^1}$ and therefore at least one contract in $Z^0_{a^1}$ must be accepted since a^1 also *qualifies for a wait time*. Due to *early filling*, the lowest available wait time contract must be accepted, which in this case is x^1, that is $x^1 \in C_m(X')$ and therefore $X^1 = \{x^1\} \subseteq C_m(X')$. In other words, the contract accepted under the described choice rule — during step 1 of the described algorithm, that is, $X^1 \subseteq \hat{C}_m(X')$ — must also be accepted under any other choice rule satisfying the described axioms $X^1 \subseteq C_m(X')$.

Induction step. We assume that if the algorithm stops at step $k-1$ then $\hat{C}_m(X') = C_m(X') = X^{k-2}$, otherwise $X^{k-1} \subseteq C_m(X')$. Given that, if the algorithm stopped at step k then $\hat{C}_m(X') = C_m(X') = X^{k-1}$, otherwise $X^k \subseteq C_m(X')$.

Note that if the algorithm stops at step $k-1$ then $\hat{C}_m(X') = C_m(X') = X^{k-2}$ by the induction assumption.

Suppose the algorithm stops at step k.

We have $\sum_{x \in X^{k-1}} s(a_x) \geq q_m$. It follows that no asylum seeker *qualifies for acceptance*, because due to the induction assumption for all remaining asylum seekers $a \in A(X^{k-1})$ we have $\sum_{\{a' \in A(C_m(X')): a' \pi_m a\}} s(a') \geq q_m$ since $X^{k-1} \subseteq C_m(X')$. It follows that $\hat{C}_m(X') = C_m(X') = X^{k-1}$ as $C_m(X')$ satisfies *respecting member state priorities*.

Suppose the algorithm does not stop at step k.

No contract in $X' \setminus Z^{k-1}$ can ever be accepted without violating *feasibility*. Hence, the only relevant asylum seekers are $A(Z^{k-1})$. As the algorithm did not stop, we have $\sum_{x \in X^{k-1}} s(a_x) < q_m$ and $Z^{k-1} \neq \varnothing$. As a^k is the highest priority asylum seeker, we have that $\sum_{\{a' \in A(C_m(X')): a' \pi_m a\}} s(a') < q_m$ and thus a^k *qualifies for acceptance*. Similarly, $|\{a' \in A(C_m(X')_{w_x}) : a' \pi a^k\}| \leq r_m^{w_x}$ holds for all $x \in Z^{k-1}_{a^k}$ and hence at least one contract in $Z^{k-1}_{a^k}$ must be accepted as a^k also *qualifies for a wait time*. Due to *early filling* the lowest available wait time contract must be accepted, which in this case is x^k, that is $x^k \in C_m(X')$ and therefore together with the induction assumption $X^k = X^{k-1} \cup \{x^k\} \subseteq C_m(X')$.

As the algorithm describing $\hat{C}_m(X')$ ends after a finite number of steps, we have $\hat{C}_m(X') = C_m(X')$.

□

Proposition 1

Proof. Consider X' and $X' \cup \{x^*\}$ for some $x^* \in X \setminus X'$. We refer to the relevant sets during each step of the algorithm for C_m as $X^{k\prime\prime}$, $Z^{k\prime\prime}$, and so on under the former ($C_m(X')$) and $X^{k\prime}$, $Z^{k\prime}$, and so on under the latter ($C_m(X' \cup \{x^*\})$).

Given that $x^* \notin C_m(X' \cup \{x^*\})$, we want to show that $C'_m(X' \cup \{x^*\}) = C_m(X')$. We proceed by induction.

Base step: We assume that $x^* \notin C_m(X' \cup \{x^*\})$. If the former algorithm ($C_m(X')$) stops at step 1 we have $C_m(X' \cup \{x^*\}) = C_m(X')$ and $X^{1\prime} = X^{1\prime\prime}$ otherwise.

Suppose the former algorithm stops at step 1.

Case 1: If $\sum_{x \in X^{0\prime\prime}} s(a_x) \geq q_m$ then $\sum_{x \in X^{0\prime}} s(a_x) \geq q_m$ as $X^{0\prime\prime} = X^{0\prime} = \varnothing$ and therefore $C_m(X' \cup \{x^*\}) = C_m(X') = \varnothing$.

Case 2: If $\sum_{x \in X^{0\prime\prime}} s(a_x) < q_m$ but $Z^{0\prime\prime} = \varnothing$ then $Z^{0\prime} \subseteq \{x^*\}$ while if $Z^{0\prime} = \{x^*\}$ we have $C'_m(X' \cup \{x^*\}) = \{x^*\}$, leading to a contradiction. Hence $Z^{0\prime} = Z^{0\prime\prime}$ and $C_m(X' \cup \{x^*\}) = C_m(X')$.

Otherwise, consider $a^{1\prime\prime}$ defined as $a \in A(Z^{0\prime\prime})$, such that $a \pi a'$ for all $a' \in A(Z^{0\prime\prime}) \setminus \{a\}$, and $x^{1\prime\prime}$ defined as $x \in Z^{0\prime\prime}_{a^{1\prime\prime}}$, such that $w_x < w_{x'}$ for all $x' \in Z^{0\prime\prime}_{a^{1\prime\prime}} \setminus \{x\}$.

Suppose by contradiction that $x^{1\prime} \neq x^{1\prime\prime}$. Note that, by assumption $x^{1\prime} \neq x^*$ and by definition, we have $x^{1\prime} \in Z^{0\prime}$. If $a_{x^{1\prime}} \neq a_{x^{1\prime\prime}}$, we reach a contradiction, as $a_{x^{1\prime}} \in A(Z^{0\prime\prime})$ and therefore there exists a higher priority asylum seeker. Similarly, given $a_{x^{1\prime}} = a_{x^{1\prime\prime}}$ but $w_{x^{1\prime}} \neq w_{a^{1\prime\prime}}$, we reach a contradiction as $x^{1\prime} \in Z^{0\prime\prime}_{a^{1\prime\prime}}$ and the lowest wait time contract is uniquely defined. Finally, since $x^{1\prime} = x^{1\prime\prime}$ and $X^{0\prime} = X^{0\prime\prime} = \varnothing$ we have $X^{1\prime} = X^{0\prime} \cup \{x^{1\prime}\} = X^{0\prime\prime} \cup \{x^{1\prime\prime}\} = X^{1\prime\prime}$.

Induction step: By the induction assumption, if the algorithm has not stopped at step $k-1$, we have $X^{k-1\prime} = X^{k-1\prime\prime}$. We want to show that if the former algorithm stops at step k we have $C_m(X' \cup \{x^*\}) = C_m(X')$ and $X^{k\prime} = X^{k\prime\prime}$ otherwise.

Suppose the former algorithm stops at step k.

Case 1: If $\sum_{x \in X^{k-1\prime\prime}} s(a_x) \geq q_m$ then $\sum_{x \in X^{k-1\prime}} s(a_x) \geq q_m$ as $X^{k-1\prime\prime} = X^{k-1\prime}$ and therefore $C_m(X' \cup \{x^*\}) = C_m(X') = X^{k\prime\prime}$.

Case 2: If $\sum_{x \in X^{k-1\prime\prime}} s(a_x) < q_m$ but $Z^{k\prime\prime} = \varnothing$ then $Z^{k\prime} =\subseteq \{x^*\}$. If $Z^{k\prime} = \{x^*\}$ we have $C'_m(X' \cup \{x^*\}) = \{x^*\}$, leading to a contradiction. Hence $Z^{k\prime} = Z^{k\prime\prime}$ and $C_m(X' \cup \{x^*\}) = C_m(X')$.

Otherwise, consider $a^{k\prime\prime}$ defined as $a \in A(Z^{k\prime\prime})$, such that $a \pi a'$ for all $a' \in A(Z^{k\prime\prime}) \setminus \{a\}$, and $x^{k\prime\prime}$ defined as $x \in Z^{k\prime\prime}_{a^{k\prime\prime}}$, such that $w_x < w_{x'}$ for all $x' \in Z^{k\prime\prime}_{a^{k\prime\prime}} \setminus \{x\}$.

Suppose by contradiction that $x^{k\prime} \neq x^{k\prime\prime}$. Note that by assumption $x^{k\prime} \neq x^*$ and moreover $Z^{k\prime} \setminus \{x^*\} = Z^{k\prime\prime}$. If $a_{x^{k\prime}} \neq a_{x^{k\prime\prime}}$, we reach a contradiction, as $a_{x^{k+1\prime}} \in A(Z^{k\prime\prime})$ and the highest priority asylum seeker is uniquely defined. Similarly, given $a_{x^{k\prime}} = a_{x^{k\prime\prime}}$ but $w_{x^{k\prime}} \neq w_{a^{k\prime\prime}}$, we reach a contradiction, as $x^{k\prime} \in Z^{k\prime\prime}_{a^{k\prime\prime}}$ and the lowest wait time contract is uniquely defined. Finally, since $x^{k\prime} = x^{k\prime\prime}$ and by the induction assumption $X^{k\prime} = X^{k\prime\prime}$, we have $X^{k\prime} = X^{k-1\prime} \cup \{x^{k\prime}\} = X^{k-1\prime\prime} \cup \{x^{k\prime\prime}\} = X^{k\prime\prime}$.

$\square$

Lemma 1

Proof. We have to show that for any $X' \subseteq X$, whenever there do not exist two contracts $x, x' \in C'_m(X')$ specifying the same asylum seeker $a(x) = a(x')$ we must have $C'_m(X') = C_m(X')$. The proof is by induction on contracts accepted during the steps of the algorithms describing the choice functions.

Base Step: If the algorithm describing C_m stops at the end of step 1 then $C'_m(X') = C_m(X')$ and $X^{1\prime} = X^1$ otherwise.

Suppose the algorithm stops at step 1.

By the definition of step 0 we have $X^{0\prime} = X^0 = \varnothing$ and therefore $Z^{0\prime} = \{x \in X'_m : r_m^{w_x} > |X^0_{w_x}| \text{ and } x \notin X^0)\} = Z^0 = \{x \in X'_m : r_m^{w_x} > |X^0_{w_x}| \text{ and } a_x \notin A(X^0)\}$. Hence, if $\sum_{x \in X^0} s(a_x) \geq q_m$, then $\sum_{x \in X^{0\prime}} s(a_x) \geq q_m$ and both algorithms end. Similarly, if $Z^0 = \varnothing$ then $Z^{0\prime} = \varnothing$ and both algorithms end. In both instances $C'_m(X') = X^{0\prime} = X^0 = C_m(X')$.

Suppose the algorithm does not stops at step 1.

Under the original choice rule the highest priority asylum seeker in $a \in A(Z^0)$ is chosen, which is identical to the highest priority asylum seeker under $A(Z^{0\prime}) = A(Z^0)$, and hence $a^{1\prime} = a^1$. Moreover, the original choice rule selects the lowest wait time contact in Z^0, which is identical to the lowest wait time contract in $Z^{0\prime} = Z^0$, and hence $x^{1\prime} = x^1$. Trivially, we have $X^{1\prime} = X^{0\prime} \cup \{x^{1\prime}\} = X^0 \cup \{x^1\} = X^1$.

Induction Step: We assume that if the algorithm describing C_m stops at the end of step $k-1$, then $C'_m(X') = C_m(X')$ and $X^{k-1\prime} = X^{k-1}$ otherwise.

We show that if the algorithm describing C_m stops at the end of step k, then $C'_m(X') = C_m(X')$ and $X^{k\prime} = X^k$ otherwise.

Suppose the algorithm stops at step k.

By the induction assumption $X^{k-1\prime} = X^{k-1}$. Hence, if $\sum_{x \in X^0} s(a_x) \geq q_m$ then $\sum_{x \in X^{0\prime}} s(a_x) \geq q_m$ and both algorithms end, and by the induction assumption $C'_m(X') = X^{k-1\prime} = X^{k-1} = C_m(X')$.

By the induction assumption we have $X^{k-1\prime} = X^{k-1}$, as well as by definition $Z^{k-1\prime} = \{x \in X'_m : r_m^{w_x} > |X^{k-1}_{w_x}| \text{ and } x \notin X^{k-1})\} \supseteq Z^{k-1} = \{x \in X'_m : r_m^{w_x} > |X^{k-1}_{w_x}| \text{ and } a_x \notin A(X^{k-1})\}$. Suppose there exist some $x \in Z^{k-1\prime} \setminus Z^{k-1}$. In this case x belongs to an asylum seeker already being assigned a contract $a_x \in A(X^{k-1})$ with higher priority than any asylum seeker $a \in A(Z^{k-1})$. In other words, $x^{k\prime} \in Z^{k-1\prime} \setminus Z^{k-1}$ and at the same time we reach a contradiction

with the initial assumption that there do not exist two accepted contracts specifying the same contracts, that is, there exists $x \in X^{k-1}$ with $a_x = a_{x^{k\prime}}$. We get $Z^{k-1\prime} = Z^{k-1}$.
Suppose $Z^{k-1} = \varnothing$. Then $Z^{k-1\prime} = Z^{k-1}$ implies that $C'_m(X') = X^{k-1\prime} = X^{k-1} = C_m(X')$.
Suppose the algorithm does not stops at step k.
Under the original choice rule the highest priority asylum seeker in $a \in A(Z^k)$ is chosen, which is identical to the highest priority asylum seeker under $A(Z^{k\prime}) = A(Z^k)$, and hence $a^{k\prime} = a^k$. Moreover, the original choice rule selects the lowest wait time contact in Z^k, which is identical to the lowest wait time contract in $Z^{k\prime} = Z^k$, and hence $x^{k\prime} = x^k$. Trivially, we have $X^{k\prime} = X^{k-1\prime} \cup \{x^{k\prime}\} = X^{k-1} \cup \{x^k\} = X^k$.

□

Lemma 2

Lemma 2 is helpful for analyzing the completion C'_m. Lemma 2 compares the contracts accepted under $C'_m(X)$ and $C'_m(X \cup \{x^*\})$ given that $x^* \in C_m(X' \cup \{x^*\})$ and under the assumption that the burden-sharing stopping condition is not binding, that is, assuming $\sum_{x \in X' \cup \{x^*\}} s(a_x) \geq q_m$. In essence, it states that there exists at most one previously accepted contract $x \in C'_m(X)$ that gets rejected $x \notin C'_m(X \cup \{x^*\})$.

Lemma 2. *Assume $\sum_{x \in X' \cup \{x^*\}} s(a_x) \geq q_m$. Consider $C'_m(X')$ and $C_m(X' \cup \{x^*\})$ for some $x^* \in X \setminus X'$ with $x^* \in C_m(X' \cup \{x^*\})$. Let $\{x^{1\prime\prime}, \dots, x^{K\prime\prime}\} = C'_m(X')$ and $\{x^{1\prime}, \dots, x^{l\prime} = x^*, \dots, x^{K\prime}\} \subseteq C'_m(X \cup \{x^*\})$. Let step ℓ_1 be the step at which x^* gets accepted. If $|C'_m(X')_{w_{x^*}}| = r^{w_{x^*}}$, let step ℓ_2 be the step at which the last contract with wait time w_{x^*} gets accepted ($\ell_2 \geq \ell_1$).*
Consider a contract $x^{j\prime} \neq x^*$

- *i) if $j < \ell_1$ then $x^{j\prime} = x^{j\prime\prime}$,*
- *ii) if $\ell_2 > j > \ell_1$ then $x^{j\prime} = x^{j-1\prime\prime}$, and*
- *iii) if $j > \ell_2$ then $x^{j\prime} = x^{j\prime\prime}$.*

Proof. Consider X' and $X' \cup \{x^*\}$ for some $x^* \in X \setminus X'$. We refer to the relevant sets during each step of the algorithm for C'_m as $X^{k\prime\prime}$, $Z^{k\prime\prime}$, and so on under the former ($C'_m(X')$) and $X^{k\prime}$, $Z^{k\prime}$, and so on under the latter ($C'_m(X' \cup \{x^*\})$).
For now, consider both algorithms but ignore the stopping point of either algorithm due to $\sum_{x \in X^{k-1\prime}} s(a_x) \geq q_m$ and $\sum_{x \in X^{k-1\prime\prime}} s(a_x) \geq q_m$, respectively. In other words, assume that $\sum_{x \in X' \cup \{x^*\}} s(a_x) \geq q_m$.

Moreover, assume that in step l the latter accepts $x^{l\prime} = x^*$ and the former $x^{l\prime\prime}$ and denote w_{x^*} by w^*. Also, set the contracts accepted under the former as $\{x^{1\prime\prime}, \dots, x^{l\prime\prime}, \dots, x^{l+k\prime\prime}, \dots, x^{K\prime\prime}\}$.
Base Step. Consider step $l+1$.
Case 1: If $|X^{l\prime\prime}_{w^*}| = |X^{l\prime}_{w^*}| = r^{w^*}_m$ and $|X^{l\prime\prime}_w| = |X^{l\prime}_w|$ for all $w \in W$, then $x^{l+1\prime} = x^{l+1\prime\prime}, \dots, x^{K\prime} = x^{K\prime\prime}$.
Case 2: If $|X^{l\prime\prime}_{w^*}| = |X^{l\prime}_{w^*}| < r^{w^*}_m$ and $|X^{l\prime\prime}_w| = |X^{l\prime}_w|$ for all $w \in W$, then $x^{l+1\prime} = x^{l\prime\prime}$ with $w^* = w_{x^{l\prime\prime}}$.
Case 3: If $|X^{l\prime\prime}_{w_{x^{l\prime\prime}}}| = |X^{l\prime}_{w_{x^{l\prime\prime}}}|+1$, $|X^{l\prime}_{w^*}| = |X^{l\prime\prime}_{w^*}|+1$, and $|X^{l\prime\prime}_w| = |X^{l\prime}_w|$ for all $w \in W \setminus \{w^*, w_{x^{l\prime\prime}}\}$, then $x^{l+1\prime} = x^{l\prime\prime}$ with $w^* \neq w_{x^{l\prime\prime}}$.
Up until step l both algorithms accept identical contracts during each step, so $X^{l-1\prime} = X^{l-1\prime\prime}$. Note that $Z^{k\prime} = \{x \in X'_m : r^{x_w}_m > |X^{l\prime}_w| \text{ and } x \notin X^{l\prime}\}$ as $X^{l\prime} = X^{l-1\prime} \cup \{x^*\}$ and $Z^{k\prime\prime} = \{x \in X'_m : r^{x_w}_m > |X^{l\prime\prime}_w| \text{ and } x \notin X^{l\prime\prime}\}$ with $X^{l\prime\prime} = X^{l-1\prime} \cup \{x^{l\prime\prime}\}$. It follows that we can consider the following three relevant cases, covering every possible outcome.
Case 1: $w^* = w_{x^{l\prime\prime}}$ and $|X^{l-1\prime\prime}_{w^*}| + 1 = |X^{l-1\prime}_{w^*}| + 1 = r^{w^*}_m$.
We have $|X^{l\prime\prime}_{w^*}| = |X^{l\prime}_{w^*}| = r^{w^*}_m$, $|X^{l\prime\prime}_w| = |X^{l\prime}_w|$ for all $w \in W$, and $X^{l\prime\prime} \setminus \{x^{l\prime\prime}\} = X^{l\prime} \setminus \{x^*\}$. Hence,

$$\begin{aligned} Z^{l\prime} &= \{x \in X'_m : r^{x_w}_m > |X^{l\prime}_w| \text{ and } x \notin X^{l\prime}\} \\ &= \{x \in X'_m : r^{x_w}_m > |X^{l\prime\prime}_w| \text{ and } x \notin X^{l\prime\prime}\} \\ &= Z^{l\prime\prime}. \end{aligned}$$

It follows that every remaining step of both algorithms is identical, that is, $x^{l+1\prime} = x^{l+1\prime\prime}, \dots, x^{K\prime} = x^{K\prime\prime}$. That is of course, everything is identical, except that $\sum_{x \in X^{l\prime}} s(a_x)$ might differ from $\sum_{x \in X^{l\prime\prime}} s(a_x)$, which is irrelevant, since by assumption $\sum_{x \in X'} s(a_x) \geq q_m$.
Case 2: $w^* = w^{l\prime\prime}$ and $|X^{l-1\prime\prime}_{w^*}| + 1 = |X^{l-1\prime}_{w^*}| < r^{w^*}_m$.
We have $|X^{l\prime\prime}_{w^*}| = |X^{l\prime}_{w^*}| < r^{w^*}_m$, $|X^{l\prime\prime}_w| = |X^{l\prime}_w|$ for all $w \in W$, and $X^{l\prime\prime} \setminus \{x^{l\prime\prime}\} = X^{l\prime} \setminus \{x^*\}$. Hence,

$$\begin{aligned} Z^{l\prime} &= \{x \in X'_m : r^{x_w}_m > |X^{l\prime}_w| \text{ and } x \notin X^{l\prime}\} \\ &= \{x \in X'_m : r^{x_w}_m > |X^{l-1\prime\prime}_w| \text{ and } x \notin X^{l-1\prime\prime}\} \\ &= Z^{l-1\prime\prime}. \end{aligned}$$

Since $x^{l\prime\prime}$ was chosen due to having the highest priority and lowest wait time in $Z^{l-1\prime\prime}$, we have $x^{l+1\prime} = x^{l\prime\prime}$.
Case 3: $w^* \neq w^{l\prime\prime}$.
We have $|X^{l\prime\prime}_{w_{x^{l\prime\prime}}}| = |X^{l\prime}_{w_{x^{l\prime\prime}}}| + 1$, $|X^{l\prime}_{w^*}| = |X^{l\prime\prime}_{w^*}| + 1$, $|X^{l\prime\prime}_w| = |X^{l\prime}_w|$ for all $w \in W \setminus \{w^*, w_{x^{l\prime\prime}}\}$,

and $X^{l''} \setminus \{x^{l''}\} = X^{l'} \setminus \{x^*\}$. Hence,

$$\begin{aligned} Z^{k'} &= \{x \in X'_m : r_m^{x_w} > |X_w^{l'}| \text{ and } x \notin X^{l'}\} \\ &\subseteq \{x \in X'_m : r_m^{x_w} > |X_w^{l-1''}| \text{ and } x \notin X^{l-1'}\} \\ &= Z^{l-1''} \end{aligned}$$

Since $x^{l''}$ was chosen due to having the highest priority and lowest wait time in $Z^{l-1''}$, we have $x^{l+1'} = x^{l''}$ as $x^{l''} \in Z^{l'} \subseteq Z^{l-1''}$.

Induction Step.

Induction assumption: Assume there are cases 1, 2, and 3 at step $l+k$.

Case 1: If $|X_{w^*}^{l+k-1''}| = |X_{w^*}^{l+k-1'}| = r_m^{w^*}$ and $|X_w^{l+k-1''}| = |X_w^{l+k-1'}|$ for all $w \in W$ then $x^{l+k'} = x^{l+k''}, \dots, x^{K'} = x^{K''}$.

Case 2: If $|X_{w^*}^{l+k-1''}| = |X_{w^*}^{l+k-1'}| < r_m^{w^*}$ and $|X_w^{l+k-1''}| = |X_w^{l+k-1'}|$ for all $w \in W$ then $x^{l+k'} = x^{l+k-1''}$, with $w^* = w_{x^{l+k-1''}}$.

Case 3: If $|X_{w_{x^{l+k-1''}}}^{l+k-1''}| = |X_{w_{x^{l+k-1''}}}^{l+1'}| + 1$, $|X_{w^*}^{l+k-1'}| = |X_{w^*}^{l+k-1''}| + 1$ and $|X_w^{l+k-1''}| = |X_w^{l+k-1'}|$ for all $w \in W \setminus \{w^*, w_{x^{l''}}\}$ then $x^{l+k'} = x^{l+k-1''}$, with $w^* \neq w_{x^{l+k-1''}}$.

There are the same cases 1, 2, and 3 at step $l+k+1$.

Case 1 holds at step $l+k$.

Note that at step $l+k$ if we are in case 1 then trivially case 1 holds for step $l+k+1$.

Case 2 holds at step $l+k$.

By the induction assumption, $|X_{w^*}^{l+k-1''}| = |X_{w^*}^{l+k-1'}| < r_m^{w^*}$,$|X_w^{l+k-1''}| = |X_w^{l+k-1'}|$ for all $w \in W$, and we have $x^{l+k'} = x^{l+k-1''}$, with $w^* = w_{x^{l+k-1''}}$.

Case 1: We are in case 1 if $w_{x^{l+k''}} = w^*$ and $|X_{w^*}^{l+k-1''}| + 1 = |X_{w^*}^{l+k-1'}| + 1 = r_m^{w^*}$.

We have $|X_{w^*}^{l+k''}| = |X_{w^*}^{l+k'}| = r_m^{w^*}$, $|X_w^{l+k''}| = |X_w^{l+k'}|$ for all $w \in W$, and $X^{l+k''} \setminus \{x^{l+k''}\} = X^{l+k'} \setminus \{x^*\}$. Hence,

$$\begin{aligned} Z^{l+k'} &= \{x \in X'_m : r_m^{x_w} > |X_w^{l+k'}| \text{ and } x \notin X^{l+k'}\} \\ &= \{x \in X'_m : r_m^{x_w} > |X_w^{l+k''}| \text{ and } x \notin X^{l+k''}\} \\ &= Z^{l+k''}. \end{aligned}$$

Therefore, all remaining accepted contracts are identical, that is $x^{l+k+1'} = x^{l+k+1''}, \dots, x^{K'} = x^{K''}$.

Case 2: We are in case 2 if $w_{x^{l+k''}} = w^*$ and $|X_{w^*}^{l+k-1''}| + 1 = |X_{w^*}^{l+k-1'}| + 1 = r_m^{w^*}$.

We have $|X_{w^*}^{l+k''}| = |X_{w^*}^{l+k'}| < r_m^{w^*}$, $|X_w^{l+k''}| = |X_w^{l+k'}|$ for all $w \in W$, and $X^{l+k''} \setminus \{x^{l+k''}\} =$

$X^{l+k\prime} \setminus \{x^*\}$. Hence,

$$\begin{aligned}Z^{l+k\prime} &= \{x \in X'_m : r_m^{x_w} > |X_w^{l+k\prime}| \text{ and } x \notin X^{l+k\prime}\}\\ &= \{x \in X'_m : r_m^{x_w} > |X_w^{l+k-1\prime\prime}| \text{ and } x \notin X^{l+k-1\prime\prime}\}\\ &= Z^{l+k-1\prime\prime}.\end{aligned}$$

Since $x^{l+k\prime\prime}$ was chosen due to having the highest priority and lowest wait time in $Z^{l-1\prime\prime}$, we have $x^{l+k+1\prime} = x^{l+k\prime\prime}$.

Case 3: We are in case 3 if $w_{x^{l+k\prime\prime}} \neq w^*$.

We have $|X_{w_{x^{l+k\prime\prime}}}^{l+k\prime\prime}| = |X_{w_{x^{l+k\prime\prime}}}^{l+k\prime}| + 1$, $|X_{w^*}^{l+k\prime}| = |X_{w^*}^{l+k\prime\prime}| + 1$, $|X_w^{l+k\prime\prime}| = |X_w^{l+k\prime}|$ for all $w \in W \setminus \{w^*, w_{x^{l+k\prime\prime}}\}$, and $X^{l+k\prime\prime} \setminus \{x^{l+k\prime\prime}\} = X^{l+k\prime} \setminus \{x^*\}$. Hence,

$$\begin{aligned}Z^{l+k\prime} &= \{x \in X'_m : r_m^{x_w} > |X_w^{l+k\prime}| \text{ and } x \notin X^{l+k\prime}\}\\ &\subseteq \{x \in X'_m : r_m^{x_w} > |X_w^{l+k-1\prime\prime}| \text{ and } x \notin X^{l+k-1\prime\prime}\}\\ &= Z^{l+k-1\prime\prime}.\end{aligned}$$

Since $x^{l+k\prime\prime}$ was chosen due to having the highest priority and lowest wait time in $Z^{l+k-1\prime\prime}$, we have $x^{l+k+1\prime} = x^{l+k\prime\prime}$ as $x^{l+k\prime\prime} \in Z^{l+k\prime} \subseteq Z^{l+k-1\prime\prime}$.

Case 3 holds at step $l + k$.

By the induction assumption, $|X_{w_{x^{l+k-1\prime\prime}}}^{l+k-1\prime\prime}| = |X_{w_{x^{l+k-1\prime\prime}}}^{l+k-1\prime}| + 1$, $|X_{w^*}^{l+k-1\prime}| = |X_{w^*}^{l+k-1\prime\prime}| + 1$, $|X_w^{l+k-1\prime\prime}| = |X_w^{l+k-1\prime}|$ for all $w \in W \setminus \{w^*, w_{x^{l\prime\prime}}\}$ and we have $x^{l+k\prime} = x^{l+k-1\prime\prime}$, with $w^* \neq w_{x^{l+k-1\prime\prime}}$.

Case 1: We are in case 1 if $w_{x^{l+k\prime\prime}} = w^*$ and $|X_{w^*}^{l+k-1\prime\prime}| + 1 = r_m^{w^*}$.

We have $|X_{w^*}^{l+k\prime\prime}| = |X_{w^*}^{l+k\prime}| = r_m^{w^*}$, $|X_w^{l+k\prime\prime}| = |X_w^{l+k\prime}|$ for all $w \in W$and $X^{l+k\prime\prime} \setminus \{x^{l+k\prime\prime}\} = X^{l+k\prime} \setminus \{x^*\}$.

The remaining argument for $x^{l+k+1\prime} = x^{l+k+1\prime\prime}, \ldots, x^{K\prime} = x^{K\prime\prime}$ is identical to the previous argument for case 1 as $Z^{l+k\prime} = Z^{l+k\prime\prime}$.

Case 2: We are in case 2 if $w_{x^{l+k\prime\prime}} = w^*$ and $|X_{w^*}^{l+k-1\prime\prime}| + 1 < r_m^{w^*}$.

We have $|X_{w^*}^{l+k\prime\prime}| = |X_{w^*}^{l+k\prime}| < r_m^{w^*}$ and $|X_w^{l+k\prime\prime}| = |X_w^{l+k\prime}|$ for all $w \in W$, and $X^{l+k\prime\prime} \setminus \{x^{l+k\prime\prime}\} = X^{l+k\prime} \setminus \{x^*\}$.

The remaining argument for $x^{l+k+1\prime} = x^{l+k\prime\prime}$ is identical to the previous argument for case 2, as $Z^{l+k\prime} = Z^{l+k-1\prime\prime}$.

Case 3: We are in case 3 if $w_{x^{l+k\prime\prime}} \neq w^*$.

We have $|X_{w_{x^{l+k\prime\prime}}}^{l+k\prime\prime}| = |X_{w_{x^{l+k\prime\prime}}}^{l+k\prime}| + 1$, $|X_{w^*}^{l+k\prime\prime}| + 1 = |X_{w^*}^{l+k\prime}|$, $|X_w^{l+k\prime\prime}| = |X_w^{l+k\prime}|$ for all $w \in W \setminus \{w^*, w_{x^{l+k\prime\prime}}\}$, and $X^{l+k\prime\prime} \setminus \{x^{l+k\prime\prime}\} = X^{l+k\prime} \setminus \{x^*\}$.

The remaining argument for $x^{l+k+1\prime} = x^{l+k\prime\prime}$ is identical to the previous argument for case 3, as $x^{l+k\prime\prime} \in Z^{l+k\prime} \subseteq Z^{l+k-1\prime\prime}$.

□

Proposition 2

Proof. This proof is almost identical to the proof for Proposition 1. Consider X' and $X' \cup \{x^*\}$ for some $x^* \in X \setminus X'$. We refer to the relevant sets during each step of the algorithm for C'_m as $X^{k\prime\prime}$, $Z^{k\prime\prime}$, and so on under the former ($C'_m(X')$) and $X^{k\prime}$, $Z^{k\prime}$, and so on under the latter ($C'_m(X' \cup \{x^*\})$).

Given that $x^* \notin C'_m(X' \cup \{x^*\})$, we want to show that $C'_m(X' \cup \{x^*\}) = C'_m(X')$. We proceed by induction.

Base step: We assume that $x^* \notin C'_m(X' \cup \{x^*\})$. If the former algorithm ($C'_m(X')$) stops at step 1, we have $C'_m(X' \cup \{x^*\}) = C'_m(X')$ and $X^{1\prime} = X^{1\prime\prime}$ otherwise.

Suppose the former algorithm stops at step 1.

Case 1: If $\sum_{x \in X^{0\prime\prime}} s(a_x) \geq q_m$, then $\sum_{x \in X^{0\prime}} s(a_x) \geq q_m$ as $X^{0\prime\prime} = X^{0\prime} = \varnothing$ and therefore $C'_m(X' \cup \{x^*\}) = C'_m(X') = \varnothing$.

Case 2: If $\sum_{x \in X^{0\prime\prime}} s(a_x) < q_m$ but $Z^{0\prime\prime} = \varnothing$, then $Z^{0\prime} \subseteq \{x^*\}$ while if $Z^{0\prime} = \{x^*\}$ we have $C'_m(X' \cup \{x^*\}) = \{x^*\}$, which leads to a contradiction. Hence $Z^{0\prime} = Z^{0\prime\prime}$ and $C'_m(X' \cup \{x^*\}) = C'_m(X')$.

Otherwise, consider $a^{1\prime\prime}$ defined as $a \in A(Z^{0\prime\prime})$, such that $a \pi a'$ for all $a' \in A(Z^{0\prime\prime}) \setminus \{a\}$, and $x^{1\prime\prime}$ defined as $x \in Z^{0\prime\prime}_{a^{1\prime\prime}}$, such that $w_x < w_{x'}$ for all $x' \in Z^{0\prime\prime}_{a^{1\prime\prime}} \setminus \{x\}$.

Suppose by contradiction that $x^{1\prime} \neq x^{1\prime\prime}$. Note that, by assumption $x^{1\prime} \neq x^*$ and by definition, we have $x^{1\prime} \in Z^{0\prime}$. If $a_{x^{1\prime}} \neq a_{x^{1\prime\prime}}$ we reach a contradiction, as $a_{x^{1\prime}} \in A(Z^{0\prime\prime})$ and therefore there exists a higher priority asylum seeker. Similarly, given $a_{x^{1\prime}} = a_{x^{1\prime\prime}}$ but $w_{x^{1\prime}} \neq w_{a^{1\prime\prime}}$, we reach a contradiction, as $x^{1\prime} \in Z^{0\prime\prime}_{a^{1\prime\prime}}$ and the lowest wait time contract is uniquely defined. Finally, since $x^{1\prime} = x^{1\prime\prime}$ and $X^{0\prime} = X^{0\prime\prime} = \varnothing$, we have $X^{1\prime} = X^{0\prime} \cup \{x^{1\prime}\} = X^{0\prime\prime} \cup \{x^{1\prime\prime}\} = X^{1\prime\prime}$.

Induction step: By the induction assumption, if the algorithm has not stopped at step $k-1$, we have $X^{k-1\prime} = X^{k-1\prime\prime}$. We want to show that if the former algorithm stops at step k, we will have $C'_m(X' \cup \{x^*\}) = C'_m(X')$ and $X^{k\prime} = X^{k\prime\prime}$ otherwise.

Suppose the former algorithm stops at step k.

Case 1: If $\sum_{x \in X^{k-1\prime\prime}} s(a_x) \geq q_m$, then $\sum_{x \in X^{k-1\prime}} s(a_x) \geq q_m$ as $X^{k-1\prime\prime} = X^{k-1\prime}$ and therefore $C'_m(X' \cup \{x^*\}) = C'_m(X') = X^{k\prime\prime}$.

Case 2: If $\sum_{x \in X^{k-1\prime\prime}} s(a_x) < q_m$ but $Z^{k\prime\prime} = \varnothing$, then $Z^{k\prime} =\subseteq \{x^*\}$. If $Z^{k\prime} = \{x^*\}$, we have $C'_m(X' \cup \{x^*\}) = \{x^*\}$, which leads to a contradiction. Hence $Z^{k\prime} = Z^{k\prime\prime}$ and $C'_m(X' \cup \{x^*\}) = C'_m(X')$.

Otherwise, consider $a^{k\prime\prime}$ defined as $a \in A(Z^{k\prime\prime})$, such that $a\pi a'$ for all $a' \in A(Z^{k\prime\prime}) \setminus \{a\}$, and $x^{k\prime\prime}$ defined as $x \in Z^{k\prime\prime}_{a^{k\prime\prime}}$ such that $w_x < w_{x'}$ for all $x' \in Z^{k\prime\prime}_{a^{k\prime\prime}} \setminus \{x\}$.
Suppose by contradiction that $x^{k\prime} \neq x^{k\prime\prime}$. Note that, by assumption $x^{k\prime} \neq x^*$ and moreover $Z^{k\prime} \setminus \{x^*\} = Z^{k\prime\prime}$. If $a_{x^{k\prime}} \neq a_{x^{k\prime\prime}}$, we reach a contradiction, as $a_{x^{k+1\prime}} \in A(Z^{k\prime\prime})$ and the highest priority asylum seeker is uniquely defined. Similarly, given $a_{x^{k\prime}} = a_{x^{k\prime\prime}}$ but $w_{x^{k\prime}} \neq w_{a^{k\prime\prime}}$, we reach a contradiction as $x^{k\prime} \in Z^{k\prime\prime}_{a^{k\prime\prime}}$ and the lowest wait time contract is uniquely defined. Finally, since $x^{k\prime} = x^{k\prime\prime}$ and by the induction assumption $X^{k\prime} = X^{k\prime\prime}$, we have $X^{k\prime} = X^{k-1\prime} \cup \{x^{k\prime}\} = X^{k-1\prime\prime} \cup \{x^{k\prime\prime}\} = X^{k\prime\prime}$.
By Lemma 1, C'_m is a completion of $C_m(X')$, and therefore there exists a completion satisfying irrelevance of rejected contracts. □

Proposition 4

Proof. Consider X' and $X' \cup \{x^*\}$ for some $x^* \in X \setminus X'$. We refer to the relevant sets during each step of the algorithm for C'_m as $X^{k\prime\prime}$, $Z^{k\prime\prime}$, and so on under the former ($C'_m(X')$) and $X^{k\prime}$, $Z^{k\prime}$, and so on under the latter ($C'_m(X' \cup \{x^*\})$).
Case 1. Suppose that $x^* \notin C'_m(X' \cup \{x^*\})$.
By irrelevance of rejected contracts $C'_m(X' \cup \{x^*\}) = C'_m(X')$ and therefore $|C'_m(X' \cup \{x^*\})| = |C'_m(X')|$.
Case 2. Suppose that $x^* \in C'_m(X' \cup \{x^*\})$.
Consider a step j with $\ell_2 \geq j > \ell_1$ where we have $X^{j-1\prime\prime} = \{x^{1\prime\prime} \dots, x^{\ell_1\prime\prime}, \dots, x^{j-1\prime\prime}\}$ and by Lemma 2 we get that $X^{j-1\prime} = \{x^{1\prime\prime} \dots, x^*, x^{\ell_1\prime\prime}, \dots, x^{j-2\prime\prime}\}$. In other words, $X^{j-1\prime\prime} \setminus \{x^{j-1\prime\prime}\} = X^{j-1\prime} \setminus \{x^*\}$ and by small burden-size priority $s(a_{x^*}) \leq s(a_{x^{j-1\prime\prime}})$ and therefore $\sum_{x \in X^{j-1\prime}} s(a_x) \leq \sum_{x \in X^{j-1\prime\prime}} s(a_x)$.
It follows that if another step is taken under X', then another step is taken under $X' \cup \{x\}$.
Similarly, for any step j with $j > \ell_2$, we have $X^{j-1\prime\prime} = \{x^{1\prime\prime} \dots, x^{\ell_1\prime\prime}, \dots, x^{\ell_2\prime\prime}, \dots, x^{j-1\prime\prime}\}$ and by Lemma 2 we get that $X^{j-1\prime} = \{x^{1\prime\prime}, \dots, x^*, x^{\ell_1\prime\prime}, \dots, x^{\ell_2\prime} = x^{\ell_2-1\prime\prime}, x^{\ell_2+1\prime} = x^{\ell_2+1\prime\prime}, \dots, x^{j-1\prime} = x^{j-1\prime\prime}\}$.
In other words, $X^{j-1\prime\prime} \setminus \{x^{\ell_2\prime\prime}\} = X^{j-1\prime} \setminus \{x^*\}$ with $a_{x^*} \pi a_{x^{\ell_2\prime\prime}}$ and by small burden-size priority $s(a_{x^*}) \leq s(x^{\ell_2\prime\prime})$ and therefore $\sum_{x \in X^{j-1\prime}} s(a_x) \leq \sum_{x \in X^{j-1\prime\prime}} s(a_x)$.
It follows that if another step is taken under X', then another step is taken under $X' \cup \{x\}$.
To sum up, under small burden-size, if another step is taken under the algorithm for X', then another step is taken under the algorithm for $X' \cup \{x\}$. Since each step corresponds to an accepted contract, we have $|C'_m(X' \cup \{x^*\})| \geq |C'_m(X')|$.
By Lemma 1, C'_m is a completion of $C_m(X')$, and therefore there exists a completion satisfying the law of aggregate demand.

□

Proposition 3

Proof. Consider X' and $X' \cup \{x^*\}$ for some $x^* \in X \setminus X'$. We refer to the relevant sets during each step of the algorithm for C'_m as $X^{k\prime\prime}$, $Z^{k\prime\prime}$, and so on under the former ($C'_m(X')$) and $X^{k\prime}$, $Z^{k\prime}$, and so on under the latter ($C'_m(X' \cup \{x^*\})$).

Case 1. Suppose that $x^* \notin C'_m(X' \cup \{x^*\})$.

By irrelevance of rejected contracts $C'_m(X' \cup \{x^*\}) = C'_m(X')$ and therefore if $x \in C'_m(X' \cup \{x^*\})$ then $x \in C'_m(X')$.

Case 2. Suppose that $x^* \in C'_m(X' \cup \{x^*\})$.

Consider a step j with $\ell_2 \geq j > \ell_1$ where we have $X^{j-1\prime\prime} = \{x^{1\prime\prime} \dots, x^{\ell_1\prime\prime}, \dots, x^{j-1\prime\prime}\}$ and by Lemma 2 we get that $X^{j-1\prime} = \{x^{1\prime\prime} \dots, x^*, x^{\ell_1\prime\prime}, \dots, x^{j-2\prime\prime}\}$.

In other words, $X^{j-1\prime\prime} \setminus \{x^{j-1\prime\prime}\} = X^{j-1\prime} \setminus \{x^*\}$ and by large burden-size priority $s(a_{x^*}) \geq s(a_{x^{j-1\prime\prime}})$ and therefore $\sum_{x \in X^{j-1\prime}} s(a_x) \geq \sum_{x \in X^{j-1\prime\prime}} s(a_x)$.

It follows that if another step is taken under $X' \cup \{x\}$, then another step is taken under X' — large-size priority reverses the observation under Proposition 4.

Similarly, for any step j with $j > \ell_2$, we have $X^{j-1\prime\prime} = \{x^{1\prime\prime} \dots, x^{\ell_1\prime\prime}, \dots, x^{\ell_2\prime\prime}, \dots, x^{j-1\prime\prime}\}$ and by Lemma 2 we get that $X^{j-1\prime} = \{x^{1\prime\prime}, \dots, x^*, x^{\ell_1\prime\prime}, \dots, x^{\ell_2\prime} = x^{\ell_2-1\prime\prime}, x^{\ell_2+1\prime} = x^{\ell_2+1\prime\prime}, \dots, x^{j-1\prime} = x^{j-1\prime\prime}\}$. In other words, $X^{j-1\prime\prime} \setminus \{x^{\ell_2\prime\prime}\} = X^{j-1\prime} \setminus \{x^*\}$ with $a_{x^*} \pi a_{x^{\ell_2\prime\prime}}$ and by large burden-size priority $s(a_{x^*}) \leq s(a_{x^{**}})$ and therefore $\sum_{x \in X^{j-1\prime}} s(a_x) \geq \sum_{x \in X^{j-1\prime\prime}} s(a_x)$.

It follows that if another step is taken under $X' \cup \{x\}$, then another step is taken under X'.

To sum up, under large burden-size, if another step is taken under the algorithm for $X' \cup \{x\}$, then another step is taken under the algorithm for X'. Therefore, if $x \in C'_m(X' \cup \{x^*\}) \setminus \{x^*\}$ then $x \in C'_m(X')$

By Lemma 1, C'_m is a completion of $C_m(X')$, and therefore there exists a completion satisfying substitutability.

□