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Keywords: Affirmative Action, Rotation, Indivisibility, Apportionment, Divisor Methods
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Affirmative Action's Connected Apportionments*

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Abstract

It is not feasible to reserve a fraction of a seat, therefore to respect the spirit of affirmative action the Indian reservation system adheres to a rotating system of seat claiming, commonly referred to as a *roster*. Developing a roster involves addressing a series of connected apportionment problems. To identify suitable apportionment methods, six essential requirements direct our search to the large class of divisor methods. Our study reveals that Webster's method is the unique divisor method that satisfies several practical and fairness properties, making it an excellent choice for constructing rosters. Our comparative analysis of the existing Indian roster with the application of Webster's method underscores the benefits of the latter approach.

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1 Introduction

Indian affirmative action scheme, *the reservation policy*, unlike its American counterpart, explicitly prescribes a proportion of seats and jobs in publicly funded institutions to various beneficiary groups. Every recruitment or admissions advertisement must include information about the proportion of government positions that are specifically designated as ”reserved” for various protected groups. Since seats are indivisible and arise in small numbers over time, innovative methods are used to help achieve the objectives of the reservation policy in practice. For instance, a university may appoint at most one assistant professor of economics every year, while the reservation policy may have five beneficiary groups. To ensure that, over a period of time, each beneficiary group gets its reservation policy prescribed percentage of seats, India devised a tool called *roster*.[1]

The publicly announced *roster* details a sequence of length two hundred in which beneficiary groups of a reservation policy take turns in claiming seats. The objective of maintaining rosters is to provide representation in proportion to reservation fractions mandated by the policy and the chosen route is rotation in claiming seats:[2]

> `Though members of a particular category in a particular recruitment year may be unlucky and may not get proportionate benefit but their lucky successors in later recruitment years may get more than what is due to them, thus, making up for the earlier deficiency and vice versa.`

The reservation policy dictates a beneficiary group's total number of turns in a roster. However, the sequence of taking turns is not fixed by the legislation and is therefore up to the designers (which are the many state governments of India). Since seats arrive over time in small numbers, the delay in claiming seats occurs naturally. However, it would be an unjustified layer of partiality if the delay is systematically associated with a beneficiary group's proportion mandated by the policy.

In this note we study the problem of constructing rosters. Section 2 presents the rosters as a solution to a series of connected apportionment problems. Section 3 shows that any serious contender for the problem among the apportionment methods must be from the large class of divisor methods. Section 4 shows that practical and fairness considerations favor Webster's method among divisor methods. Furthermore, Theorem 2 and Theorem 3 give two new properties of Webster's method. Section 5 scrutinizes the Indian roster while contrasting it with Webster's method's roster. The paper concludes with a discussion with respect to the related literature in Section 6. Proofs are relegated to Appendix B.

[1]For India's roster, visit `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`, last accessed on 26 June 2023.

[2]See page 10 of `https://www.police.rajasthan.gov.in/Rajpolice/pdf_files/2462008_155643_Reservation.pdf`, last accessed on 26 June 2023.

2 Formulation

A **roster construction problem** is a tuple $\Lambda = (\mathcal{C}, \boldsymbol{\alpha}, n)$. $\mathcal{C}$ is a finite set of **categories** where $m := |\mathcal{C}| \geq 2$. The **reservation policy** is defined by a vector of fractions $\boldsymbol{\alpha} = (\alpha_j)_{j \in \mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0,1)$ fraction of turns are to be reserved so that $\sum_{j \in \mathcal{C}} \alpha_j = 1$. The **size of the roster** is n where n is a positive integer. Throughout the paper, we fix a set of categories $\mathcal{C}$ and a reservation policy $\boldsymbol{\alpha}$.

A **roster** $R_n : \{1, \ldots, n\} \to \mathcal{C}$ maps each position to a category such that, $|R_n^{-1}(j)| = \alpha_j n$ for all $j \in \mathcal{C}$. We denote the set of rosters by $\mathcal{R}_n$. The definition incorporates the idea that the total number of positions a roster assigns to a category is the same as the proportion given by the reservation policy; that is, all categories must get their quantum of reserved positions once n positions are filled. Therefore, we can restrict our attention to only those rosters where this is possible.

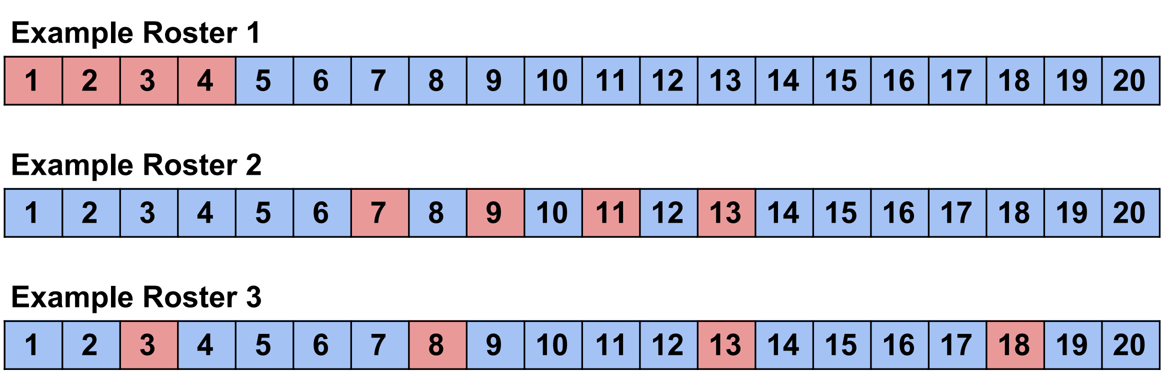
We denote by x_j^t the number of seats given to category j until position t under roster R_n; that is, $x_j^t := |\{s \in R_n^{-1}(j) \mid s \leq t\}|$. We denote by n_j the total number of seats given to category j; that is, $n_j := \alpha_j n$. In line with the practice of making rosters, we assume that n is chosen such that $n_j \in \mathbb{N}$, that is, the total number of turns for each category is a natural number.

We next introduce an example that makes the notion of roster easier to comprehend. There are two categories for easy illustration. The example will also be sufficient to present the various aspects of designing rosters in upcoming sections.

Example 1. *Consider a problem $\Lambda = (\{R,\ B\}, \boldsymbol{\alpha} = [0.2, 0.8], 20)$. There are two categories $\mathcal{C} = \{R,\ B\}$, represented by red and blue colors. The reservation policy reserves 20% positions for members of category R. The size of the roster is $n = 20$. Therefore, the number of positions given to the category R and B is 4 and 16, respectively. Figure 1 illustrates three possible rosters for the problem. For instance, Example Roster 1 is*

$$R_n(k) = \begin{cases} R, & \text{if } k \in \{1,2,3,4\} \\ B, & \text{otherwise} \end{cases}$$

Figure 1: EXAMPLE ROSTERS



The following representation of a roster entails a staircase where each step is some category's turn. The **staircase representation of roster** R_n is

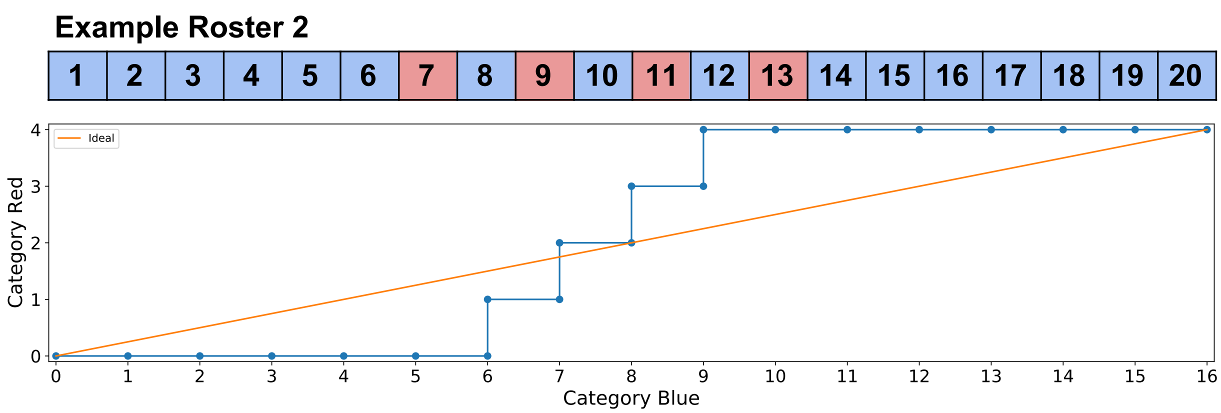
$$\mathbf{x^t} = (x_1^t, \ldots, x_j^t, \ldots, x_m^t), \quad \text{for} \quad t \in \{1, \ldots, n\},$$

where $x_j^t := |\{s \in R_n^{-1}(j) \mid s \leq t\}|$ measures the number of seats category j receives until position t. Figure 2 illustrates staircase representation for two rosters depicted in Figure 1.

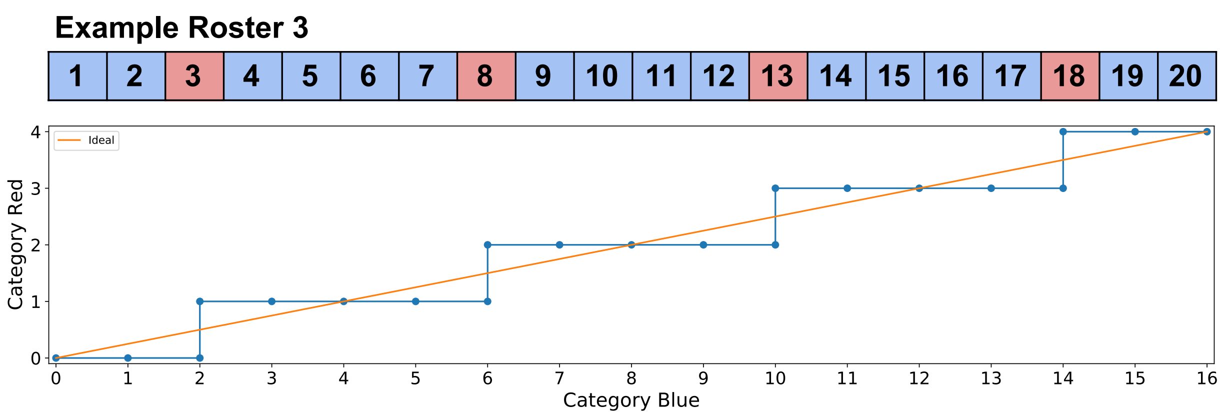
Denote the **standard unit vector in the direction of the** j**-th axis** by $\mathbf{e_j}$; that is, vector with j-th component equals 1 and all other components equal 0. Given two consecutive points $\mathbf{x^{t-1}}$ and $\mathbf{x^t}$, if $\mathbf{x^t} = \mathbf{x^{t-1}} + \mathbf{e_j}$, then we say that staircase **moves to direction** j **at step** t; that is, $x_j^t = x_j^{t-1} + 1$. Note that for staircase representation of a roster, $\mathbf{x^t}$ can move in only one direction at any step.

The roster construction problem may therefore be seen as a series of connected apportionment problems, where each step t of the staircase $\mathbf{x^t} = (x_1^t, \ldots, x_m^t)$ is an apportionment of t seats among m categories with $\mathbf{q^t} = (q_j^t)_{j \in \mathcal{C}} := (\alpha_1 t, \ldots, \alpha_m t)$ claims. Notice that step t corresponds to *house size* and claim q_j^t corresponds to *quota* in the original model of apportionment (see Balinski and Young (2001)).

Figure 2: STAIRCASE REPRESENTATION



(a) EXAMPLE ROSTER 2



(b) EXAMPLE ROSTER 3

3 Methods

A **method of apportionment** is a point to set mapping Φ that assigns at least one solution $\mathbf{x^t}$ to each $\mathbf{q^t}$. Numerous methods have been proposed and utilized to address the problem at hand (see Young (1995), Balinski and Young (2001), Pukelsheim (2017)). To grasp the characteristics of their solutions, it is crucial to examine the properties they adhere to. Among these properties, three are unquestionably essential and have been satisfied by every method ever seriously proposed:

- A method of apportionment Φ is **anonymous** if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ implies $\sigma(\mathbf{x^t}) \in \Phi(\sigma(\mathbf{q^t}))$ for any permutation σ of the categories. A category should receive the same number of seats wherever it may appear in the list of categories.
- A method of apportionment Φ is **exact** if $\mathbf{q^t}$ is integer-valued implies $\Phi(\mathbf{q^t}) = \mathbf{x^t}$. If perfect proportionality may be achieved, it must be.
- A method of apportionment Φ is **responsive** if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ and $q_i^t > q_j^t$ implies $x_i^t \geq x_j^t$. Category with a relatively higher claim should never receive fewer seats.

A basic principle of fair apportionment (Balinski (2005)) is that: "Any part of a fair apportionment must be fair." To define this idea, let $\mathbf{x}_S^t := \left(x_j^t\right)_{j\in S}$, $\mathbf{q}_S^t := \left(q_j^t\right)_{j\in S}$ and $x(S) := \sum_{j\in S} x_j^t$, for $S \subset \mathcal{C}$.

- A method of apportionment Φ is **consistent** if $\mathbf{x}^t \in \Phi\left(\mathbf{q}^t\right)$ then $\mathbf{x}_S^t \in \Phi\left(\mathbf{q}_S^{x(S)}\right)$ for any subset of categories $S \subset \mathcal{C}$; moreover, if a sub-problem has another solution $\mathbf{z}_S^{x(S)} \in \Phi\left(\mathbf{q}_S^{x(S)}\right)$ another solution to the problem itself exists, $\left(\mathbf{z}_S^{x(S)}, \mathbf{x}_{(\mathcal{C}\setminus S)}^t\right) \in \Phi\left(\mathbf{q}^t\right)$.

Two other natural properties for apportionment methods to be used for roster construction problem:

- A method of apportionment Φ is **house-monotone** if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ implies there is some $\mathbf{x^{t+1}} \in \Phi(\mathbf{q^{t+1}})$ for which $\mathbf{x^{t+1}} \geq \mathbf{x^t}$. Reservations are irreversible, going from step t to step $t+1$ in a roster, the number of position each category receives upto each step can only weakly increase.
- A method of apportionment Φ is **balanced** if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ and $q_i^t = q_j^t$ implies $|x_i^t - x_j^t| \leq 1$. If two categories have the same reservation fractions, and therefore the same claims, their apportionments should not differ by more than one seat.

Following Balinski and Ramirez (2014), a **divisor function** d is any monotone real-valued function defined on the nonnegative integers satisfying $d(k) \in [k, k+1]$ for any integer k, and for which there exists no pair of integers $a \geq 0$ and $b \geq 1$ with $d(a) = a+1$ and $d(b) = b$.

The **divisor method** Φ^d based on d is

$$\Phi^d(\mathbf{q^t}) = \left\{\mathbf{x^t} : \min_{x_i^t > 0} \frac{q_i^t}{d\left(x_i^t - 1\right)} \geq \max_{x_j^t \geq 0} \frac{q_j^t}{d\left(x_j^t\right)}, x(\mathcal{C}) = \sum_{j\in\mathcal{C}} x_j^t = t\right\}$$

A divisor method in decreasing order the values ${}^{q_i^t}/_{d(x_i^t)}$ for all i and all integer a_i give the priority by which category i receives its x_i^tth seat. Any divisor method is consistent, house-monotone and balanced in addition to satisfying the three essential properties (see Balinski and Young (2001), Pukelsheim (2017)). Arguably the most important result in the theory of apportionment that characterizes divisor methods provides further affirmations:

Theorem 1. *(Balinski and Young, 2001, p. 147) A method of apportionment Φ is consistent, responsive, exact and anonymous if and only if it is a divisor method Φ^d.*

4 Why Webster?

The gist of the previous section is that any method of apportionment seriously worth an investigation must be divisor method. The question then arises: Which of the infinite number of divisor methods should be chosen?

This section contends that one particular member stands out – Webster's method that requires $d(a) = a + \frac{1}{2}$ (also known as Sainte-Laguë method or the major fractions method).

A roster R_n is a **Webster's staircase** if for all $\mathbf{x^t}$ and $t \in \{1, \ldots, n\}$ we have,

$$\min_{x_i^t > 0} \frac{\alpha_i}{x_i^t - 0.5} \geq \max_{x_j^t \geq 0} \frac{\alpha_j}{x_j^t + 0.5},$$

which is the min-max inequality that characterizes Webster apportionments.

4.1 Concatenation Invariance

Note the following two principles that are followed for maintenance of rosters:[3]

```
(f) The register / roster register shall be maintained in the form
    of a running account year after year. For example if recruitment
    in a year stops at point 6, recruitment in the following year
    shall begin from point 7.
(h) In case of cadres where reservation is given by rotation, fresh
    cycle of roster shall be started after completion of all the
    points in the roster.
```

Therefore, the roster not only decides the allocation of seats $1, 2, \ldots, n$, but also the allocation of seats $n + 1, \ldots, 2n$, the allocation of seats $2n + 1, \ldots, 3n$, and so forth. A roster pins down an allocation of an infinite sequence of seats constructed as concatenation of infinitely many finite seat sequences of length n. Our next property requires that the apportionment method must be invariant to such concatenation. That is, a roster of size kn can be constructed by concatenation of k copies of a roster of size n. For example, among rosters depicted in Figure 1, only example roster 3 is invariant to such concatenation.

Given roster construction problem Λ, let s denote the size of the smallest roster possible, that is, let s be the lowest common denominator of the reservation fractions.

[3]See page 1 of https://dopt.gov.in/sites/default/files/Ch-05_2014.pdf, last accessed on 26 June, 2023.

- A method of apportionment Φ is **concatenation invariant** if $t \in \mathbb{N}_+$, $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$, and $\mathbf{x}^s \in \Phi(\mathbf{q}^s)$ implies $\mathbf{x}^t + \mathbf{x^s} \in \Phi(\mathbf{q}^{t+s})$.

Not all divisor methods are concatenation invariant, but Webster's method is.[4]

Theorem 2. *Webster's method is a concatenation invariant divisor method.*

4.2 Equitable Treatment in a Roster

Since rosters entails a series of connected apportionments, the set of positions at which a category is at advantage and the distribution of these sets must be further analyzed to determine the goodness of the methods and to differentiate or classify them. In addition to the previous argument in favor of using Webster's method, we next argue that that rosters generated solely through the application of Webster's method treat beneficiaries in an equitable manner.

Since seats are indivisible, at every position in the roster, between any two categories, there will always be a certain partiality that gives one of the categories a slight advantage over the other. The distribution of seats for a category would make this advantage clear in the case of rosters. The **cumulative distribution of seats for category** j under roster R_n is

$$F_j(t) := \frac{|\{s \in R_n^{-1}(j) \ | \ s \leq t\}|}{|R_n^{-1}(j)|} = \frac{x_j^t}{n_j}, \quad \text{for} \quad t \in \{1, \dots .n\}.$$

These cumulative distribution functions measure the fraction of seats a category receives until position t; that is, the number of seats given to a category until position t over the total number of seats given to a category. For instance, Figure 3 illustrates the cumulative distribution of seats for the rosters depicted in Figure 1.

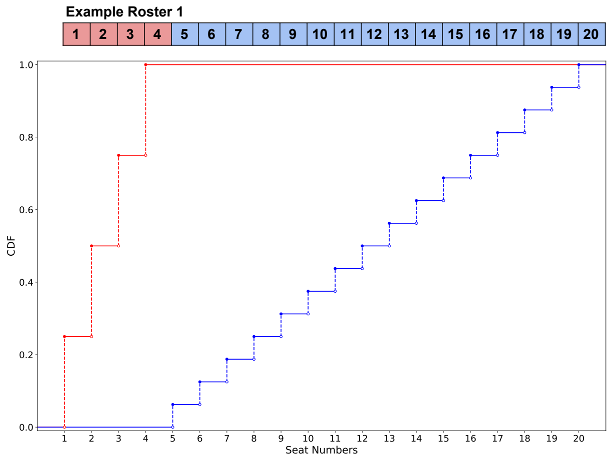
4.2.1 Spreads seats as evenly as possible

Had the seats been divisible, the uniform distribution would be the ideal seat allocation for equitable treatment of categories. In that case, the seats would be spread "as evenly as possible" without favoring any category over the other at any point in the roster, thus treating all categories as equally as possible. However, since seats are indivisible, at each step some category is over-represented, while another is under-represented. As a measure of the grievances of each category, deviations from the uniform distribution is a reasonable measure of partiality.

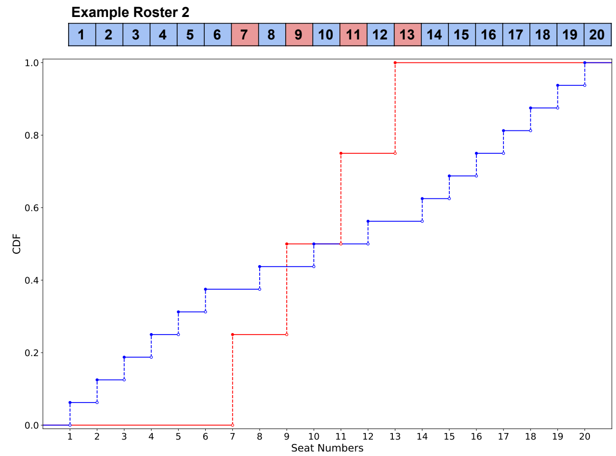
We denote by $U(t)$ uniform distribution; that is, for any $t \in \{1, \dots, n\}$, $U(t) = t/n$. In evaluating a roster's disuniformity at position t we consider $|F_j(t) - U(t)|$ the distance between the distribution of seats for category j and the uniform distribution, which is squared, weighted by the claim α_j, before summing across categories. This leads to a **disuniformity index** at step t defined as

[4] An example of a divisor method that is not concatenation invariant is Hill's method $d(a) = \sqrt{a(a+1)}$. Consider example 1, $\Lambda = (\{R, \ B\}, \boldsymbol{\alpha} = [0.2, 0.8], 20)$, and notice that $s = 5$. To see why Hill's method is not concatenation invariant, consider $t = 2$. $(1,1) \in \Phi((0.4, 1.2))$, $(1,4) \in \Phi((1,4))$, but $(2,5) \notin \Phi((1.4, 5.6))$, instead only $(1,6) \in \Phi((1.4, 5.6))$.

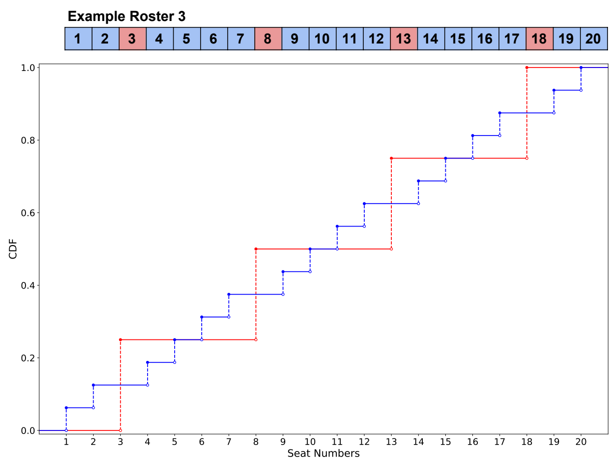
Figure 3: DISTRIBUTION OF SEATS



(a) EXAMPLE ROSTER 1



(b) EXAMPLE ROSTER 2



(c) EXAMPLE ROSTER 3

$$\mathrm{DI}(\mathbf{x^t}) = \sum_{j \in \mathcal{C}} \alpha_j (F_j(t) - U(t))^2.$$

- A method of apportionment Φ **minimizes disuniformity** if $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$ implies $\mathrm{DI}(\mathbf{x^t}) \leq \mathrm{DI}(\mathbf{x^t} + \mathbf{e_j} - \mathbf{e_i})$ for all categories $i, j \in \mathcal{C}$.

Theorem 3. *Webster's method is the unique divisor method that minimizes disuniformity.*

Uniformity hints at proportionality, and Webster's method of apportionment hints at Sainte-Laguë index of proportionality (Lijphart and Gibberd, 1977, p. 241). We next show how these two are related.

Consider the ratio between the seat allocation x_j^t and claim q_j^t for each category j and position t, in particular x_j^t/q_j^t. In a perfectly proportional outcome, $x_j^t/q_j^t = 1$ for each j and

t. In evaluating a roster's disproportionality at position t we consider $\left|\frac{x_j^t}{q_j^t}-1\right|$ as error term for each category j, which is squared, weighted by the claim q_j^t, before summing across categories. This leads to the well-known **Sainte-Laguë index** at step t defined as

$$\mathrm{SLI}(\mathbf{x^t}) = \sum_{j\in\mathcal{C}} q_j^t \left(\frac{x_j^t}{q_j^t}-1\right)^2 = \sum_{j\in\mathcal{C}} \frac{(x_j^t-q_j^t)^2}{q_j^t}.$$

It can easily be shown that the disunifomity index and the Sainte-Laguë index of disproportionality are co-monotone. In particular, we have the following relationship.

Lemma 1. $\mathrm{DI}(\mathbf{x^t}) = {}^t\!/_{n^2}\ \mathrm{SLI}(\mathbf{x^t})$.

4.2.2 Minimizes inequality

Since seats are indivisible, at every position in the roster, between any two categories, there will always be a certain partiality that gives one of the categories a slight advantage over the other. It is straightforward to say that for any pair of categories $i, j \in \mathcal{C}$, category i is **favored relative** to category j at position t under roster R_n, if $F_i(t) > F_j(t)$. One measure of inequality therefore is $|F_i(t) - F_j(t)|$. For example, in Figure 3, Example Roster 3 has lesser inequality compared to Example Roster 2 at all positions. Huntington (1928) writes that "such a transfer should be made or not depends on whether the amount of inequality between the two states after the transfer is less or greater than it was before; if the amount of inequality is reduced by the transfer, it is obvious that the transfer should be made."

- A method of apportionment Φ **minimizes inequality** if $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$ implies $|{}^{x_i^t}\!/_{n_i} - {}^{x_j^t}\!/_{n_j}| \leq |{}^{(x_j^t+1)}\!/_{n_j} - {}^{(x_i^t-1)}\!/_{n_i}|$ for all categories $i, j \in \mathcal{C}$.

Theorem 4. *(Balinski and Young, 2001, p. 101) Webster's method is the unique divisor method that minimizes inequality.*

4.2.3 Stays near quota

In line with the concept of Pareto optimality, at any given stage in the roster, it should be impossible to transfer a seat from one category to another in a way that brings both categories' seat allocations closer to their respective claims. Put simply, it is not feasible to bring one category closer to its claim without simultaneously moving another category further away from its claim.

- A method of apportionment Φ **stays near quota** if $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$ implies there are no categories $i, j \in \mathcal{C}$ such that $|(x_i^t-1) - q_i^t| < |x_i^t - q_i^t|$ and $|(x_j^t+1) - q_j^t| < |x_j^t - q_j^t|$.

Theorem 5. *(Balinski and Young, 2001, p. 132) Webster's method is the unique divisor method that stays near quota.*

5 The Roster Made in India

$\Lambda^{\texttt{IN}} = (\{UR, OBC, SC, EWS, ST\}, \boldsymbol{\alpha} = [0.405, 0.27, 0.15, 0.10, 0.075], 200)$ is the Indian roster construction problem. It consists of five categories of seats – Unreserved (UR), Other Backward Classes (OBC), Scheduled Castes (SC), Economically Weaker Sections (EWS), and Scheduled Tribes (ST). The reservation policy dictates the division of the 200 seats in a roster among the five categories – 81 *UR*, 54 *OBC*, 30 *SC*, 20 *EWS*, and 15 *ST*. However, the positions each category is assigned in a roster is left up to the designer, in this case the Ministry of Personnel.[5] In all central government institutions, rosters are constructed and maintained as per the most recent revision detailed in *Office memorandum No.36039/1/2019-Estt (Res)* dated January 31, 2019 issued by the Department of Personnel and Training (Ministry of Personnel, Public Grievances, and Pensions, Government of India).[6]

It is hard to improve upon the transparency rosters provide in the nationwide implementation of the reservation policy. Moreover, they do achieve the goal of reserving seats in a manner such that each institution meets the prescribed percentage of reserved over a sufficiently long period of time. Yet the choice of the roster construction method can and has been scrutinized for the delay associated with the arrival of reserved seats. Gupta and Thorat (2019) and Gupta (2018) criticize the current roster method for delaying reserved seats, causing a sparse representation of some reserved category candidates. They write:

> `A mathematical juggling has been used by policymakers to reduce the constitutionally mandated reservation for the deprived sections.`

The complaint becomes apparent on analyzing how the distribution of seats is systematically associated with a beneficiary group's reservation fraction. Recall that, for any pair of categories $i, j \in \mathcal{C}$, category i is favored relative to category j at position t under roster R_n, if $F_i(t) > F_j(t)$. One measure of partiality is to count such instances. Let $\#|F_i(t) > F_j(t)| := |\{t \in \{1, \ldots, n\} | F_i(t) > F_j(t)\}|$ denote the number of positions category i is favored relative to category j. In Table 1, for $\alpha_i < \alpha_j$ we note the value of $\#|F_i(t) < F_j(t)| - \#|F_i(t) > F_j(t)|$, call it pairwise bias. In pairwise comparisons, this measures whether the roster tends to exhibit a greater frequency of favoring categories with a larger proportion over the other, surpassing the instances where the opposite occurs.

Table 1: Instances of Pairwise Bias

	ST vs. EWS	ST vs. SC	ST vs. OBC	ST vs. UR	EWS vs. SC	EWS vs. OBC	EWS vs. UR	SC vs. OBC	SC vs. UR	OBC vs. UR
Indian Roster	57	100	146	197	54	126	195	86	198	199
Webster's Staircase	3	0	3	3	-4	2	1	6	9	-3

Table 1 shows systematic association between the distribution of seats and the reservation fractions. (i) the Indian roster favors categories with a larger reservation fraction relative

[5] The details of the method for making rosters can be found in the *Annexure I to Office Memorandum No. 36012/2/96-Estt(Res)* dated July 2, 1997. Visit `https://documents.doptcirculars.nic.in/D2/D02adm/OM%20dated%202%207%2097BsMyq.pdf`, last accessed on 26 June 2023.

[6] For *Office memorandum No.36039/1/2019-Estt (Res)* visit `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`, last accessed on 26 June 2023.

to the smaller ones, and (ii) the larger the difference between reservation fractions of two categories, the higher the pairwise bias. For instance, compare category ST ($\alpha_{ST} = 0.075$), which has the smallest reservation fraction, with categories EWS ($\alpha_{EWS} = 0.1$) and UR ($\alpha_{UR} = 0.405$) which have the larger reservation fractions. The pairwise bias is 57 for ST vs. EWS, while 197 for ST vs. UR.

Table 1 also shows that such systematic delay in arrival of seats of categories with relatively smaller reservation fractions does not arise under Webster's Staircase roster. In a Webster's staircase, seats are spread "as evenly as possible" without favoring any category at a bulk of points in the roster, thus treating all categories as equally as possible.

6 Discussion

A considerable number of recent studies have documented unnoticed issues in implementation of nation-wide affirmative action policies, and have offered practical alternatives for better implementation of such policies (see Hafalir et al. (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Aygün and Turhan (2017), Dur et al. (2019), Aygun and Bó (2021), Sönmez and Yenmez (2022), and the articles cited therein). Ours is another paper in this class. While the focus of the contemporary market design literature has been the design and analysis of assignment mechanisms given reserved seats and quotas, our paper (also Evren and Khanna (2022)) looks at another side of affirmative action schemes: *proportional distribution of indivisible seats*.

The idea of rosters is similar to that of precedence orders according to which institutions prioritize individual slots above others (as in Kominers and Sönmez (2016)), but in a world where seats arrive sequentially, over time in small numbers. A serious limitation of using rosters is that it is not possible to differentiate between vertical reservations, horizontal reservations, or any form of reservations (see Sönmez and Yenmez (2022)). For instance, a roster cannot allocate all positions of a beneficiary group at the very end just as it is done in the static implementation of vertical reservations in India. On top of that, the treatment of a horizontal reservation is not any different than that of a vertical reservation in a roster.

To accommodate a richer variety of forms of reservations – vertical reservations, horizontal reservations, horizontal reservations within vertical reservations – a deviation from rosters is required. Dynamic implementation of the reservation policy is recommended, that allows reserving a seat early in time, but also allows that particular seat to be considered unreserved at a later point depending on the history of seat allocation (see Aygün and Turhan (2020)). The advantage of this method over rosters is obvious; the mandated vertical and horizontal reservations can be implemented at any point in time. However, it is hard to quash the use of rosters. Both the legislators and their electorates greatly value the ease, transparency, and credibility publicly declared rosters provide in the implementation of the reservation policy.

References

Aygün, Orhan and Bertan Turhan, "Large-scale affirmative action in school choice: Admissions to IITs in India," *American Economic Review*, 2017, *107* (5), 210–13.

_ **and** _ , "Dynamic reserves in matching markets," *Journal of Economic Theory*, 2020, *188*, 105069.

Aygun, Orhan and Inácio Bó, "College admission with multidimensional privileges: The Brazilian affirmative action case," *American Economic Journal: Microeconomics*, 2021, *13* (3), 1–28.

Balinski, Michel, "What is just?," *The American Mathematical Monthly*, 2005, *112* (6), 502–511.

_ **and Victoriano Ramirez**, "Parametric vs. divisor methods of apportionment," *Annals of Operations Research*, 2014, *215*, 39–48.

Balinski, Michel L and H Peyton Young, *Fair representation: meeting the ideal of one man, one vote*, Brookings Institution Press, 2001.

Dur, Umut, Parag Pathak, and Tayfun Sönmez, "Explicit vs. Statistical Preferential Treatment in Affirmative Action: Theory and Evidence from Chicago's Exam Schools." forthcoming," *Journal of Economic Theory*, 2019.

Echenique, Federico and M Bumin Yenmez, "How to control controlled school choice," *American Economic Review*, 2015, *105* (8), 2679–94.

Ehlers, Lars, Isa E Hafalir, M Bumin Yenmez, and Muhammed A Yildirim, "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*, 2014, *153*, 648–683.

Evren, Haydar and Manshu Khanna, "Affirmative Action in Two Dimensions: A Multi-Period Apportionment Problem," 2022.

Gupta, Anish, "The math problem in faculty reservation," *The Hindu*, 2018. Available at https://www.thehindu.com/opinion/op-ed/the-math-problem-in-faculty-reservation/article24156695.ece, last accessed on 27 July 2022.

Gupta, Anisha and Amit Thorat, "Why Are the Reserved Categories Objecting to the 13-point Roster?," *Economic & Political Weekly*, 2019.

Hafalir, Isa E, M Bumin Yenmez, and Muhammed A Yildirim, "Effective affirmative action in school choice," *Theoretical Economics*, 2013, *8* (2), 325–363.

Huntington, Edward Vermilye, "The apportionment of representatives in Congress," *Transactions of the American Mathematical Society*, 1928, *30* (1), 85–110.

Kominers, Scott Duke and Tayfun Sönmez, "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 2016, *11* (2), 683–710.

Lijphart, Arend and Robert W Gibberd, "Thresholds and payoffs in list systems of proportional representation," *European Journal of Political Research*, 1977, *5* (3), 219–244.

Pukelsheim, Friedrich, *Proportional representation*, Springer, 2017.

Sönmez, Tayfun and M. Bumin Yenmez, "Affirmative Action in India via Vertical, Horizontal, and Overlapping Reservations," *Econometrica*, 2022, *90* (3), 1143–1176.

Young, H Peyton, *Equity: in theory and practice*, Princeton University Press, 1995.

Appendices for Online Publication

A Tables and Figures

Figure 4: MODEL ROSTER OF RESERVATION WITH REFERENCE TO POSTS

FOR DIRECT RECRUITMENT

Model Roster of Reservation with reference to posts for Direct recruitment on All India Basis by Open Competition

Sl. No. of Post	Share of Entitlement				Category for which the posts should be earmarked
	SC @15%	ST @7.5%	OBC @27%	EWS @10%	
1	0.15	0.08	0.27	0.10	UR
2	0.30	0.15	0.54	0.20	UR
3	0.45	0.23	0.81	0.30	UR
4	0.60	0.30	1.08	0.40	OBC-1
5	0.75	0.38	1.35	0.50	UR
6	0.90	0.45	1.62	0.60	UR
7	1.05	0.53	1.89	0.70	SC-1
8	1.20	0.60	2.16	0.80	OBC-2
9	1.35	0.68	2.43	0.90	UR
10	1.50	0.75	2.70	1.00	**EWS-1**
11	1.65	0.83	2.97	1.10	UR
12	1.80	0.90	3.24	1.20	OBC-3
13	1.95	0.98	3.51	1.30	UR
14	2.10	1.05	3.78	1.40	ST-1
15	2.25	1.13	4.05	1.50	SC-2
16	2.40	1.20	4.32	1.60	OBC-4
17	2.55	1.28	4.59	1.70	UR
18	2.70	1.35	4.86	1.80	UR
19	2.85	1.43	5.13	1.90	OBC-5
20	3.00	1.50	5.40	2.00	SC-3
21	3.15	1.58	5.67	2.10	**EWS-2**
22	3.30	1.65	5.94	2.20	UR
23	3.45	1.73	6.21	2.30	OBC-6
24	3.60	1.80	6.48	2.40	UR
25	3.75	1.88	6.75	2.50	UR
26	3.90	1.95	7.02	2.60	OBC-7
27	4.05	2.03	7.29	2.70	SC-4
28	4.20	2.10	7.56	2.80	ST-2
29	4.35	2.18	7.83	2.90	UR
30	4.50	2.25	8.10	3.00	OBC-8
31	4.65	2.33	8.37	3.10	**EWS-3**

Source: https://dopt.gov.in/sites/default/files/ewsf28fT.PDF[7]

[7]Last accessed on 27 July 2022.

B Proofs

Let $x_j^t \coloneqq |\{s \in R_n^{-1}(j) \mid s \leq t\}|$ be the number of seats given to category j until point t. Note that $\sum_j x_j^t = t$ for $t \in \{1, \dots, n\}$. Let n_j be the total number of seats in the roster for category j, that is, $n_j = x_j^n = \alpha_j n$.

With these definitions, the distribution of seats for category j at point t is

$$F_j(t) = \frac{x_j^t}{n_j}$$

and the location at the staircase at point t is

$$\mathbf{x^t} = (x_1^t, \dots, x_j^t, \dots, x_n^t).$$

Proof of Theorem 2

Proof. $\mathbf{x}^t \in \Phi^d(\mathbf{q}^t)$ if and only if $x_j^t \geq 0$ for all $j \in \mathcal{C}$, $\sum_{j \in \mathcal{C}} x_j^t = t$, and

$$\min_{x_i^t > 0} \frac{q_i^t}{d(x_i^t - 1)} \geq \max_{x_j^t \geq 0} \frac{q_j^t}{d(x_j^t)}$$

Equivalently,

$$\frac{d(x_i^t - 1)}{d(x_j^t)} \leq \frac{\alpha_i}{\alpha_j} \leq \frac{d(x_i^t)}{d(x_j^t - 1)} \qquad \text{if } x_i^t, x_j^t > 0, \tag{1}$$

$$\frac{\alpha_i}{\alpha_j} \leq \frac{d(0)}{d(x_j^t - 1)} \qquad \text{if } x_i^t = 0, \quad x_j^t > 0, \tag{2}$$

and

$$\frac{d(x_i^t - 1)}{d(0)} \leq \frac{\alpha_i}{\alpha_j} \qquad \text{if } x_i^t > 0, \quad x_j^t = 0. \tag{3}$$

Next, since divisor methods are exact (as defined in Section 3), $\mathbf{x}^s \in \Phi^d(\mathbf{q}^s)$ we have,

$$\frac{x_i^s}{x_j^s} = \frac{\alpha_i}{\alpha_j} \tag{4}$$

Recall the following fact: adding a to the numerator and b to the denominator moves the resultant fraction closer to the fraction a/b. If $x/y < a/b$, moving the starting fraction close to a/b will make it bigger. If $x/y > a/b$, moving the starting fraction close to $\frac{a}{b}$ will make it smaller.

Adding x_i^s to numerators and x_j^s to denominators in equation (1) to (3) therefore does not alter the inequalities and gives,

$$\frac{d(x_i^t - 1) + x_i^s}{d(x_j^t) + x_j^s} \leq \frac{\alpha_i}{\alpha_j} \leq \frac{d(x_i^t) + x_i^s}{d(x_j^t - 1) + x_j^s} \qquad \text{if } x_i^t, x_j^t > 0, \tag{5}$$

$$\frac{\alpha_i}{\alpha_j} \leq \frac{d(0) + x_i^s}{d(x_j^t - 1) + x_j^s} \quad \text{if } x_i^t = 0, \quad x_j^t > 0, \tag{6}$$

and

$$\frac{d(x_i^t - 1) + x_i^s}{d(0) + x_j^s} \leq \frac{\alpha_i}{\alpha_j} \quad \text{if } x_i^t > 0, \quad x_j^t = 0. \tag{7}$$

If $d(x_i^t - 1) + x_i^s = d(x_i^t + x_i^s - 1)$, $d(x_i^t) + x_i^s = d(x_i^t + x_i^s)$, and $d(0) + x_j^s = d(x_j^s)$, then we have that $\mathbf{x}^t + \mathbf{x}^s \in \Phi^d\left(\mathbf{q}^{t+s}\right)$, that is Φ^d is concatenation invariant. In particular, Webster's method $d(a) = a + \frac{1}{2}$ is is concatenation invariant.

■

Proof of Theorem 3

We denote by $\mathrm{DI}(\mathbf{x^t})$ **weighted distance between the distribution of seats and the uniform distribution at point** t; that is,

$$\mathrm{DI}(\mathbf{x^t}) = \sum_j \alpha_j (F_j(t) - U(t))^2.$$

Note that,

$$\begin{aligned}
\mathrm{DI}(\mathbf{x^t}) &= \sum_j \alpha_j (F_j(t) - U(t))^2 \\
&= \sum_j \alpha_j \left(\frac{x_j^t}{\alpha_j n} - \frac{t}{n}\right)^2 \\
&= \frac{1}{n^2} \sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{1}{n^2} \sum_j 2x_j^t t + \frac{1}{n^2} \sum_j \alpha_j t^2 \\
&= \frac{1}{n^2} \sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{2t^2}{n^2} + \frac{t^2}{n^2} \\
&= \frac{1}{n^2} \sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{t^2}{n^2}.
\end{aligned}$$

Therefore, minimizing $\mathrm{DI}(\mathbf{x^t})$ is equivalent to minimizing $\sum_j \frac{{x_j^t}^2}{\alpha_j}$.

Lemma 2. *For any* $\mathbf{x^t} = (x_1^t, \ldots, x_j^t, \ldots, x_m^t)$, *for any pair of* i, j *with* $x_i^t > 0$,

$$DI(\mathbf{x^t}) \leq DI(\mathbf{x^t} + e_j - e_i) \iff \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}.$$

Proof. Using the fact that $\mathrm{DI}(\mathbf{x^t}) = \frac{1}{n^2}\sum_j \frac{{x_j^t}^2}{\alpha_j} - \frac{t^2}{n^2}$,

$$\begin{aligned}
\mathrm{DI}(\mathbf{x^t}) \leq \mathrm{DI}(\mathbf{x^t} + e_j - e_i) \quad &\iff \quad \frac{{x_i^t}^2}{\alpha_i} + \frac{{x_j^t}^2}{\alpha_j} \leq \frac{(x_i^t - 1)^2}{\alpha_i} + \frac{(x_j^t + 1)^2}{\alpha_j} \\
&\iff \quad \frac{{x_i^t}^2}{\alpha_i} - \frac{(x_i^t - 1)^2}{\alpha_i} \leq \frac{(x_j^t + 1)^2}{\alpha_j} - \frac{{x_j^t}^2}{\alpha_j} \\
&\iff \quad \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}.
\end{aligned}$$

■

Lemma 3. *The following sets are equivalent*

1.

$$\arg\min_{\mathbf{x^t}} \mathit{DI}(\mathbf{x^t}) \ \textit{s.t.} \ \sum_j x_j^t = t \ \textit{and} \ \mathbf{x^t} \geq 0 \ \textit{integer}$$

2.

$$\{\mathbf{x^t} \mid \ \textit{for any } i, j \textit{ with } x_i^t > 0, \ \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}; \sum_j x_j^t = t \ \textit{and} \ \mathbf{x^t} \geq 0 \ \textit{integer} \ \}$$

Proof. Lemma 2 implies that if $\mathbf{x^t}$ minimizes $\mathrm{DI}(\mathbf{x^t})$ then the following inequalities hold.

$$\frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j} \text{ for any } i, j \text{ with } x_i^t > 0$$

Therefore, if $\mathbf{x^t}$ is in the former set, then it is also in the latter set.

Suppose $\mathbf{y^t}$ is in the latter set but not in the former; that is,

$$\frac{y_i^t - 0.5}{\alpha_i} \leq \frac{y_j^t + 0.5}{\alpha_j} \text{ for any } i, j \text{ with } y_i^t > 0 \ .$$

We denote by $H = \{j | x_j^t > y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has more number of seats. We denote by $L = \{j | x_j^t < y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has less number of seats. We denote by $h_j := x_j^t - y_j^t$ for $j \in H$. We denote by $l_j := y_j^t - x_j^t$ for $j \in L$. Since $\sum_j y_j^t = \sum_j x_j^t = t$, we have $\sum_{j \in H} h_j = \sum_{j \in L} l_j > 0$.

Using the inequalities for $\mathbf{x^t}$ and $\mathbf{y^t}$ for $i \in L$ and $j \in H$, we have

$$\frac{2y_i^t - l_j}{\alpha_i} \leq \frac{2y_j^t + h_j}{\alpha_j} \text{ for any } i \in L \text{ and } j \in H \ .$$

If we calculate the summation of such inequities, we find that

$$\sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \leq \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j}.$$

Note that,

$$\begin{aligned}
\sum_j \frac{{x_j^t}^2}{\alpha_j} - \sum_j \frac{{y_j^t}^2}{\alpha_j} &= \sum_j \left(\frac{{x_j^t}^2}{\alpha_j} - \frac{{y_j^t}^2}{\alpha_j}\right) \\
&= \sum_{j \in H} \left(\frac{{x_j^t}^2}{\alpha_j} - \frac{{y_j^t}^2}{\alpha_j}\right) - \sum_{j \in L} \left(\frac{{y_j^t}^2}{\alpha_j} - \frac{{x_j^t}^2}{\alpha_j}\right) \\
&= \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j} - \sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \\
&\geq 0.
\end{aligned}$$

This contradicts the assumption that $\mathbf{y^t}$ does not belong to the first set; that is, if a $\mathbf{y^t}$ is in the second set then $\mathbf{y^t}$ must be in the first set. ■

Proof of Lemma 1

Proof.

$$\begin{aligned}
\mathrm{DI}(\mathbf{x^t}) &= \sum_{j \in \mathcal{C}} \alpha_j (F_j(t) - U(t))^2 \\
&= \sum_{j \in \mathcal{C}} \alpha_j \left(\frac{x_j^t}{\alpha_j n} - \frac{t}{n}\right)^2 \\
&= \sum_{j \in \mathcal{C}} \alpha_j \left(\frac{x_j^t - \alpha_j t}{\alpha_j n}\right)^2 \\
&= \frac{t}{n^2} \sum_{j \in \mathcal{C}} \frac{(x_j^t - \alpha_j t)^2}{\alpha_j t} \\
&= \frac{t}{n^2} \mathrm{SLI}(\mathbf{x^t}).
\end{aligned}$$

■

C Visualizing Webster's Staircase for $m = 2$

In the staircase representation, the line connecting origin $(0, 0)$ and (n_1, n_m) would be the proportionate allotment of turns if only the seats were divisible. We therefore call the line described by the following vector, the **ideal fractional line**, $\mathbf{u} =< n_1, n_m >$. For instance, in Figure 2, the ideal fractional line is the line connecting origin and $(16, 4)$.

In the staircase representation, the **euclidean distance between point $\mathbf{x^t}$ and the ideal fractional line**, is defined as the shortest distance between point $\mathbf{x^t}$ and any point on the ideal fractional line. It is the length of the line segment that is perpendicular to the ideal fractional line and passes through the point $\mathbf{x^t}$; that is,

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) = ||\mathbf{x^t} - (\mathbf{x^t} \cdot \mathbf{u})\frac{\mathbf{u}}{||\mathbf{u}||_2^2}||_2$$

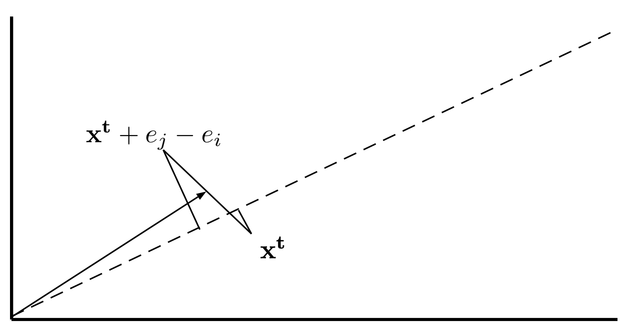
where $|| \cdot ||_2$ is the Euclidean norm.

The following iterative algorithm finds the roster that stays closest to the ideal fractional line throughout the roster. The algorithm has two parts. At each step t, for each staircase, it first finds the set of directions the staircase can move such that the Euclidean distance between the next point $\mathbf{x^t}$ and the ideal fractional line is minimum. The staircase next moves in such directions and results in a set of staircases. The algorithm creates a set of staircases in each step. Each staircase in the last step $\mathcal{S}_n$ represents a roster. We call such rosters as **Webster's Staircase rosters** because of the following result.

Lemma 4. *For any* $\mathbf{x^t} = (x_i^t, x_j^t)$ *with* $x_i^t > 0$,

$$d_{stair}(\mathbf{x^t}, \mathbf{u}) \leq d_{stair}(\mathbf{x^t} + e_j - e_i, \mathbf{u}) \iff \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}.$$

Figure 5: LEMMA 1 ILLUSTRATION



Proof. If distance from $\mathbf{x^t}$ to $\mathbf{u}$ is smaller than the distance from $\mathbf{x^t} + e_j - e_i$ to $\mathbf{u}$, then the middle point of these two points, $\mathbf{x^t} + 0.5e_j - 0.5e_i$ lays on the same side of $\mathbf{u}$ as the point $\mathbf{x^t} + e_j - e_i$ (notice the two similar triangles in fig. 5). Consequently, the slope of the line connecting origin and $(x_i^t - 0.5, x_j^t + 0.5)$ is higher than the slope of $\mathbf{u}$, $\frac{n_j}{n_i}\left(= \frac{\alpha_j}{\alpha_i}\right)$, that is,

$$\frac{x_j^t + 0.5}{x_i^t - 0.5} \geq \frac{\alpha_j}{\alpha_i}.$$

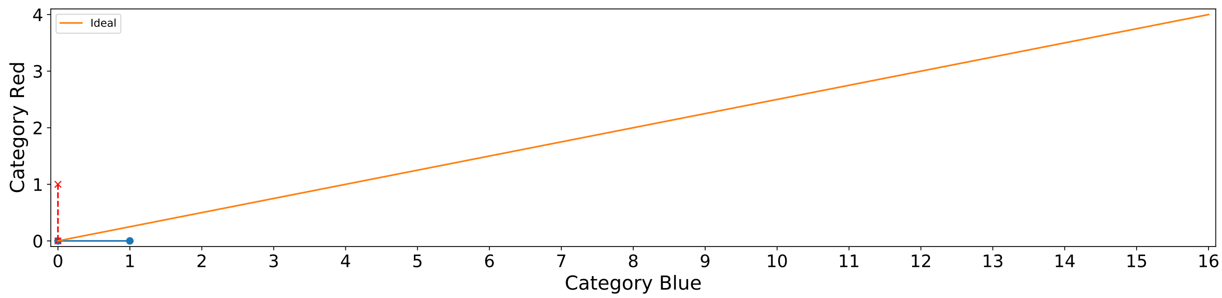
■

Webster's Staircase Generating Algorithm:

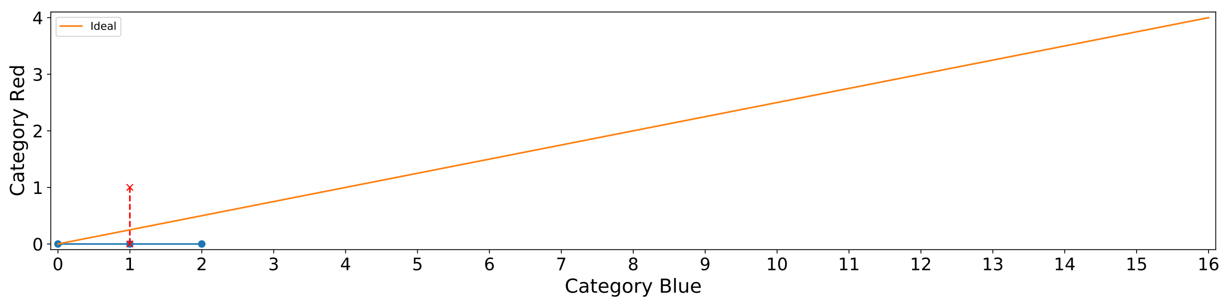
```
Input: roster construction problem Λ
Output: Webster's Staircase
S_0 = {x^0 = (0, ..., 0)};
for t ∈ {1, ..., n} do
    S_t = ∅;
    while x^{t−1} ∈ S_{t−1} do
        J_t = arg min_{j∈C} d_stair(x^{t−1} + e_j, u);
        while j_t ∈ J_t do
            x^t = x^{t−1} + e_{j_t};
            S_t = S_t ∪ {x^0, ..., x^t};
        end
    end
end
return S_n
```

To make the Webster's Staircase Generating Algorithm easier to understand and show the whole procedure that constructs the set of rosters, consider the roster construction problem Example 1. The number of positions given to the category R and B is 4 and 16, respectively. The category B is represented by the x-axis (1st axis), and the category R is represented by the y-axis (2nd axis). The ideal fractional line is the line connecting the origin and $(16, 4)$. This line is described by the vector $\mathbf{u} =< 4, 16 >$. We start from $\mathcal{S}_0 = \{\{\mathbf{x^0} = (0, \dots, 0)\}\}$. Figure 6 illustrates steps 1 to 4 of the Webster's Staircase Generating Algorithm. At step 1, staircase moves to right (direction 1) since $d_{stair}((1, 0), \mathbf{u}) < d_{stair}((0, 1), \mathbf{u})$. We add $\{(0, 0), (1, 0)\}$ to $\mathcal{S}_1$. At Step 2, staircase moves to right (direction 1) since $d_{stair}((2, 0), \mathbf{u}) < d_{stair}((1, 1), \mathbf{u})$. We add $\{(0, 0), (1, 0), (2, 0)\}$ to $\mathcal{S}_2$. At Step 3, staircase moves to up (direction 2) since $d_{stair}((2, 1), \mathbf{u}) < d_{stair}((3, 0), \mathbf{u})$. We add $\{(0, 0), (1, 0), (2, 0), (2, 1)\}$ to $\mathcal{S}_3$. At Step 4, staircase moves to right (direction 1) since $d_{stair}((3, 1), \mathbf{u}) < d_{stair}((2, 2), \mathbf{u})$. We add $\{(0, 0), (1, 0), (2, 0), (2, 1), (3, 1)\}$ to $\mathcal{S}_4$. Figure 2 (b) illustrates the final output of the algorithm.

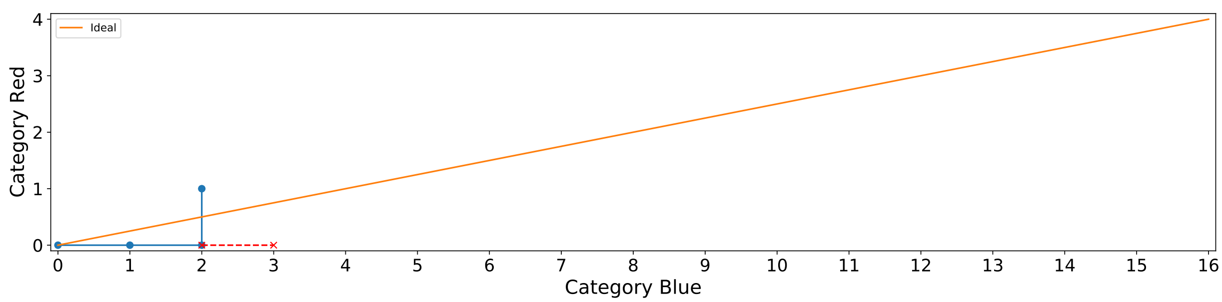
Figure 6: CONSTRUCTING Webster's STAIRCASE



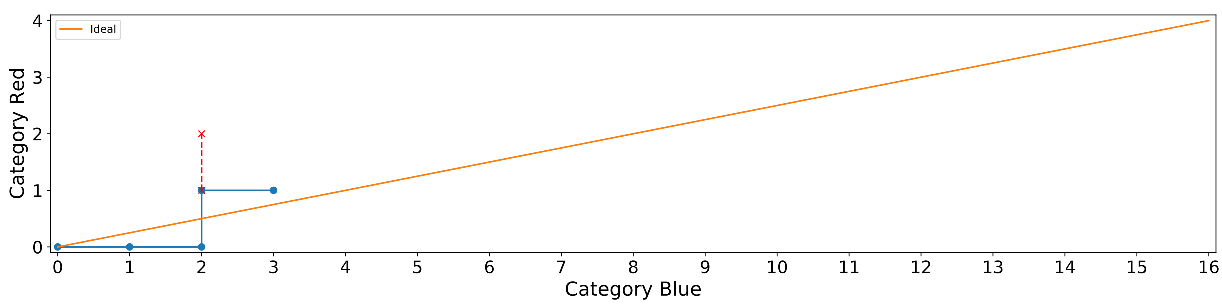
(a) STEP 1



(b) STEP 2



(c) STEP 3



(d) STEP 4