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Affirmative Action in Two Dimensions: A Multi-Period Apportionment Problem*

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In many settings, affirmative action policies apply simultaneously at two levels, for instance, at university and its departments. We show that commonly used methods in reserving positions for beneficiaries of affirmative action often need to be revised in such settings. We present a comprehensive evaluation of existing procedures to document their shortcomings. We propose a new solution with appealing theoretical properties and quantify the benefits of adopting it using recruitment advertisement data from India.

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1 Introduction

In many countries, affirmative action policies take the form of reserved seats or positions for which only eligible candidates compete. For instance, in India, beneficiary groups are entitled to their proportion of reserved seats in government jobs and publicly funded institutions (Sönmez and Yenmez (2022)). However, the policy-prescribed percentage of seats can rarely be met in practice because of the indivisible nature of positions. The fractional seats that arise in literal calculation need to be adjusted to yield whole numbers. The question then arises what the ideal whole number counterparts of an affirmative action policy prescribed fractional seats are.

Achieving policy-prescribed proportions is even more challenging when positions are heterogeneous. For instance, a university's faculty positions (say assistant professors) are listed under various departments. Each faculty position simultaneously represents two units, a department and the university. If both the departments and the university adhere to the affirmative action policy, both must reserve the prescribed percentage of seats. In this paper, we ask, how many seats should the departments and the university reserve in such cases? We term this problem as the problem of *reservations in two dimensions*.

An ideal solution to the problem of reservations in two dimensions should ensure that in each period, as well as over time, the seat allocations stay "close" to the prescribed fractional seats both (i) at the department level, and (ii) at the university level. However, delivering this is not easy. In fact the solutions seen in practice in India fail to do so. Not surprisingly, these solutions are met with several petitions and protests leading to subsequent and frequent changes in the law. The noteworthy debates from Indian courts to public arenas that inspired us to write this paper are summarized in Section 2.

The problem with existing solutions is that they do not account for the interdependence of the departments and the university in calculating reserved seats. That is, each solution either operates at the department level or the university level, but not at both simultaneously. The debate in India revolves around whether the individual departments should follow the solution or the university as a whole should follow it. If the departments follow the solution, it fails to deliver the benefit of reservations at the university level. Whereas if the university as a whole follows the solution, the reserved positions could get allocated to merely a few departments in the university. The existing solutions and their various shortcomings are formally documented in Section 4.

The aim of this paper is twofold. The first objective is to comprehensively evaluate existing solutions in light of staying within the quota property and the multi-period considerations. We do so theoretically in Section 4 and empirically in Section 6. The second objective is to check whether a solution exists that satisfactorily deals with the problem of reservations in two dimensions. That is a solution that stays "close" to the prescribed fractional seats both (i) at the department level and (ii) at the university level.

Reservations in two dimensions give rise to matrix problems, with input data as a fair share table X. Its entries x_{ij} signify the fraction of seats beneficiary j is entitled in department i as per the affirmative action policy. The rows represent the first subdivision of the university into departments. The columns accommodate several beneficiaries and therefore present a second subdivision. The university is assumed to be broken down, either way, providing department sizes as row sums and overall (university-level) beneficiary claims as column sums. The task is to find a two-way apportionment, with seat allocations (whole numbers, not fractions!) $\bar{x}_{ij}$ summing row-wise to the pre-specified row sums, while remaining "as near as may be" to the fractional seats x_{ij}.

The fair share table would be the ideal seat allocations if only the seats were divisible. Therefore, it is natural to consider integral seat allocations with entries that are rounded to an adjacent integer of entries of the fair share table as an ideal solution. That is, ideal seat allocations $\bar{x}_{ij}$ would consist of entries x_{ij} of the fair share table rounded up or down to the nearest integer. This is one of the most appealing and natural apportionment ideas, known as *staying within the quota* (see Balinski and Young (2010)). The problem of reservations in two dimensions can therefore be viewed as a rounding problem of translating a matrix of fair shares to a matrix of seat allocations obtained by rounding the fair shares up or down.

Such matrix problems are not unique to the implementation of affirmative action policies. Implementation of random and therefore fractional assignments solves such problems in the presence of a rich *bihierarchical* structure on the set of constraints (Budish et al. (2013), Pycia and Ünver (2015), and Akbarpour and Nikzad (2020)). Biproportional apportionment methods introduced by Balinski and Demange (1989a,b) deal with such problems while translating electoral votes into parliamentary seats. Controlled rounding procedures introduced by Cox and Ernst (1982) also deal with matrix problems in maintaining anonymity of census data. What is unique about the problem we analyze in the affirmative action context is their multi-period aspect. For example, a department with only one new faculty position every year must reserve the position for a different beneficiary group each year. In such cases, to ensure that each beneficiary group gets its prescribed percentage of positions over time, the beneficiary groups must take turns claiming positions. Matrix problems with such multi-period considerations are unique to reservations in two dimensions.

Our first results that deal with the problem of reservations in two dimensions without the multi-period considerations are straightforward. The rounding problem has an elegant solution, called *controlled rounding*, that stays within quota and is simple enough to be implemented by hand. The technique was introduced by Cox (1987) to make slight perturbations in two-dimensional census data to ensure the confidentiality of aggregate statistics while maintaining a good approximation of the original data. Adaptation of Cox's controlled rounding technique to our problem is summarized in Appendix A. In addition to providing a solution that stays within quota, Cox's controlled rounding procedure provides an *unbiased* lottery solution, that is, entries of the fair share matrix are rounded up or down so that ex-ante positive and negative deviations from the fair shares balance to yield no deviation from the fair shares.

The main theoretical contributions of our article address the multi-period problem of reservations in two dimensions and are presented in Section 5.2. We show that there does not exist a solution for the problem of reservations in two dimensions that stays within quota at both the university and the department level simultaneously (Proposition 2). We give an even stronger result: There does not exist a solution for which the reservation table deviates from the fair share table bounded by a finite number (Proposition 3). These results justify the struggle in figuring out a solution in real-life practice, as discussed in Section 2. Since the two constraints that staying within quota property imposes cannot be satisfied simultaneously, we ask: can these constraints be satisfied *approximately*? By approximately, we mean, the probability of violating that constraint exponentially decreases with the size of the constraint. The answer is affirmative.

The main results of the article stated in Theorem 1 and Theorem 2, show that there exists an unbiased solution that stays within quota at the department level and approximately stays within quota at the university level. The proof of Theorem 1 involves constructing a lottery solution that stays within quota at the department level and is unbiased at both the department and the university levels. An overview of the proof is presented in Section 5.2. The essential technique is to design a procedure that takes the fractions of reservations and generates a roster that lists the number of positions reserved for every number of total positions. For a roster, staying within quota constraints regulates the cumulative number of positions for each category. Since there could be many rosters that would stay within quota, the procedure generates a *random roster* by assigning each solution roster a probability. Our solution to the problem of reservation in two dimensions assigns a roster to each department adhering to the probabilities dictated by the procedure.

The procedure of constructing a random roster is built around a network flow algorithm that takes a flow network as input and randomly constructs another flow network with fewer fractional flows as its output. By iterative application of this algorithm, a flow network with integral flows is generated. The random flow network has the following two properties: the expected value of each flow after the next iteration is the same as its current value, and each constraint (imposed by the stay within quota property) remains satisfied. Since each flow network with integral flows can be mapped to a roster, this procedure generates a random roster. We next show that the approximation errors are small in Theorem 2. We do so by applying the multiplicative form of Chernoff concentration bounds to our solution to prove that, in addition to staying within quota at the department level, the solution approximately stays within quota at the university level. Moreover, we show that our bounds on the approximation errors are tight.

Lastly, in Section 6, we present an empirical case study of a two-dimensional reservations problem from India using recruitment advertisement data. The objective is twofold. The first objective is to document the shortcomings of existing procedures empirically. In particular, we highlight the severity of the problem by documenting the instances and magnitude of violations. Our second objective is to quantify the performance of our proposed solutions. We do so by running simulations on the recruitment data, thus creating reservation tables per the procedures advocated in this article and comparing the outcomes with existing (advertised) solutions.

1.1 Contributions with respect to the Related Literature

A considerable number of recent studies have offered practical alternatives for better implementation of nationwide affirmative action policies (see Abdulkadiroğlu and Sönmez (2003), Kojima (2012), Hafalir et al. (2013), Ehlers et al. (2014), Echenique and Yenmez (2015), Aygün and Turhan (2017), Dur et al. (2019), Aygun and Bó (2021), Sönmez and Yenmez (2022); Sönmez and Yenmez (2019) among others). Ours is another paper in this class. While the focus of the contemporary market design literature has been the design and analysis of assignment mechanisms given reserved seats and quotas, our paper (also Evren and Khanna (2022)) looks at another practical issue in implementation of affirmative action schemes: *proportional distribution of indivisible seats*.

Proportional distribution of indivisible objects among a group of claimants in proportion to their claims, known as the *apportionment problem*, is the center point of the seminal work of Young (1995) and Balinski and Young (2010). The two-dimensional version, the *biproportional apportionment problem*, gives rise to similar matrix problems as ours but has been investigated in the context of translating electoral votes into parliamentary seats (Gassner (1988), Balinski and Demange (1989a), Balinski and Demange (1989b), Maier et al. (2010), Lari et al. (2014)).[1] In that context, the multi-period constraints do not feature. The multi-period considerations make a reservation in two dimensions a unique apportionment problem that demands a new search for methods ensuring proportional representation.

Lastly, our paper is related to the literature on rounding techniques. The *controlled rounding* procedure introduced in Cox (1987) suffices to solve the problem of reservations in two dimensions for a particular case of the model (see Proposition 1). For the general case, our rounding approach is similar to the ones developed in the literature on *approximation algorithms* from computer science (Ageev and Sviridenko (2004), Gandhi et al. (2006) and others). These techniques are not new to market designers. The literature on the implementation of random and, therefore, fractional assignments solve such problems in the presence of a rich *bihierarchical* structure on the set of constraints (Budish et al. (2013), Pycia and Ünver (2015), and Akbarpour and Nikzad (2020)). In particular, Budish et al. (2013) and Akbarpour and Nikzad (2020) build implementation methods for random allocation mechanisms based on techniques from deterministic and randomized rounding developed in Edmonds (2003) and Gandhi et al. (2006). In addition to following a bihierarchical structure, our constraints also extend in the time dimension to accommodate the multi-period considerations. It is this multi-period aspect of our problem that renders existing solutions inadequate. A rounding procedure for a multi-period model with a bihierarchical constraint structure (upper and lower quotas at the department level and approximate constraints at the university level) is a theoretical contribution of our paper (Theorem 1 and Theorem 2).

[1]See Pukelsheim (2017) for detailed results and insights on proportional apportionment problems.

2 Motivating Debate from India

The 1950 Constitution of India provides a clear basis for positive discrimination in favor of disadvantaged groups, in the form of *reservation policies.* India's reservation policies mandate exclusive access to a fixed percentage of government jobs and seats in publicly funded institutions to the members of Scheduled Castes (SC, 15%), Scheduled Tribes (ST, 7.5%), Other Backward Classes (OBC, 27%) and Economically Weaker Sections (EWS, 10%). For transparency, the number of reserved seats for each category are explicitly and publicly advertised in advance of any admissions or recruitment cycle.

The procedures used to calculate the number of reserved seats in various settings are also explicit and public. However, they have nowhere been more contentious than in the case of universities. Unlike other government jobs, the eligibility and selection criteria change with the department for the same faculty position in a university (say, assistant professor). Thus the faculty positions in different departments are not interchangeable across a university. Each faculty position, therefore, simultaneously represents two units, a department, and the university, where each unit is subject to the reservation policy. This feature of faculty positions led to complications that made all three arms of the Indian government – the executive, the judiciary, and the legislative – intervene.

The Executive. In August 2006, the University Grants Commission (UGC) issued *Guidelines for Strict Implementation of Reservation Policy of the Government in Universities* to all government educational institutions in India.[2][3] Through these guidelines, the UGC prohibited the practice of treating *department as the unit* for application of the reservation scheme, that is, for calculating the proportion of seats to be reserved (see clause 6(c) in the guidelines). Instead, UGC mandated *university as the unit* for reservation. That is, the

[2]UGC is a statutory autonomous organization responsible for the implementation of the policy of the Central Government in the matter of admissions as well as recruitment to the teaching and non-teaching posts in central universities, state universities and institutions which are deemed to be universities.

[3]Document last accessed on 30 December 2022 at https://www.ugc.ac.in/pdfnews/7633178_English.pdf

positions in a university shall be clubbed together across departments as three separate categories: professors, associate professors (or readers), and assistant professors (or lecturers), for the application of the rule of reservation (see clause 8(a)(v) in the guidelines). However, UGC's order was challenged in court.

The Judiciary. In April 2017, the Allahabad High Court allowed a petition demanding reservations in faculty positions treating the department as the unit and quashed clauses 6(c) and 8(a)(v) of the UGC Guidelines of 2006.[4] The court argued that treating the university as the unit "would be not only impracticable, unworkable but also unfair and unreasonable" for the following two reasons stated in the judgment:

> Merely because Assistant Professor, Reader, Associate Professor, and Professor of each subject or the department are placed on the same pay-scale, but their services are neither transferable nor they are in competition with each other. It is for this reason also that clubbing of the posts for the same level treating the University as a 'Unit' would be completely unworkable and impractical. It would be violative of Article 14 and 16 of the Constitution.
>
> If the University is taken as a 'Unit' for every level of teaching and applying the roster, it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates. Such proposition again would be discriminatory and unreasonable. This, again, would be violative of Article 14 and 16 of the Constitution.

Following the court order, universities advertised vacancies with a sharp fall in the number of reserved positions. This is apparent in the case of Banaras Hindu University, presented in Table 1, where the number of unreserved seats increased from 1188 under the government's quashed solution to 1562 under the court's proposed solution.[5] The reason was that many departments had a small number of faculty positions (fewer than six). Given that each department followed the same fixed sequence in which categories take turns in claiming a

[4] Judgement last accessed on 30 December 2022 at https://indiankanoon.org/doc/177500970/

[5] Last accessed on 30 December 2022 at https://indianexpress.com/article/explained/hrd-ministry-ordinance-teacher-quota-university-prakash-javadekar-5616157/

Table 1: NUMBER OF RESERVED POSITIONS IN BANARAS HINDU UNIVERSITY

	University as a Unit (Government's Solution)					Department as a Unit (Court's Solution)				
Position	General	SC	ST	OBC	Total	General	SC	ST	OBC	Total
Professor	197	38	18	0	253	250	3	0	0	253
Associate Professor	410	79	39	0	528	500	25	3	0	528
Assistant Professor	581	172	86	310	1149	812	91	26	220	1149
Total	1188	289	143	310	1930	1562	119	29	220	1930

Notes: Data shared in government's Special Leave Petition filed in the Supreme Court of India.

position, the court's solution led to a small number of positions for the reserved categories at the university level.[6] This sparked a series of teachers' unions-led protests across India.

The Legislative. The protests compelled the government to file a petition in the Supreme Court against the Allahabad High Court verdict. "How can the post of professor of Anatomy be compared with the professor of Geography? Are you clubbing oranges with apples?" questioned the Supreme Court rejecting the appeal and terming the Allahabad high court judgment as "logical".[7] Facing a huge aggrieved vote bank, three days prior to announcement of Lok Sabha election, in March 2019, the government promulgated an ordinance that considered the university as the unit. This ordinance is now an Act of Parliament and, therefore the law in India.[8]

Today, the university is the unit for application of the reservation scheme. The court's objection that "it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates" inspired us to write this paper.

[6] See Figure 17 and Figure 18 for the sequence in which the beneficiary groups take turns in claiming a position in India.

[7] Last accessed on 30 December 2022 at https://main.sci.gov.in/supremecourt/2019/5495/5495_2019_Order_27-Feb-2019.pdf

[8] Last accessed on 30 December 2022 at http://egazette.nic.in/WriteReadData/2019/206575.pdf

3 Model and the Primitives

In this section, we formulate the problem of reservation in two dimensions. Since our primary application is the reservation of teaching positions in Indian universities, the terminology used is appropriate for that application.

3.1 Model

A **problem of reservation in two dimensions** in period $t \in \{1, 2, \dots, T\}$ is a quadruple $\Lambda^t = (\mathcal{D}, \mathcal{C}, \boldsymbol{\alpha}, (\mathbf{q}^s)_{s=1}^t)$. $\mathcal{D}$ and $\mathcal{C}$ are finite sets of **departments** and **categories** where $m := |\mathcal{D}| \geq 2$ and $n := |\mathcal{C}| \geq 2$. The **reservation scheme** is defined by a vector of fractions $\boldsymbol{\alpha} = [\alpha_j]_{j \in \mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0, 1)$ fraction of vacancies are to be reserved so that $\sum_{j \in \mathcal{C}} \alpha_j = 1$. $\mathbf{q}^s = [q_i^s]_{i \in \mathcal{D}}$ represents the **vector of vacancies** associated with the departments in period $s \in \{1, 2, \dots, t\}$. Let $Q_i^t := \sum_{s \leq t} q_i^s$ denote **period-t cumulative sum of vacancies in department i**.

A **period-t fair share table** for problem Λ^t is a two-way table

$$X^t = \begin{array}{c|c} (x_{ij}^t)_{m \times n} & (x_{i,n+1}^t)_{m \times 1} \\ \hline (x_{m+1,j}^t)_{1 \times n} & (x_{m+1,n+1}^t)_{1 \times 1} \end{array}$$

with rows indexed by $i \in \mathcal{D} \cup \{m+1\}$ and columns by $j \in \mathcal{C} \cup \{n+1\}$, such that internal entries $x_{ij}^t = \alpha_j Q_i^t$ for all $i \in \mathcal{D}$ and $j \in \mathcal{C}$, row total entries $x_{i,n+1}^t = Q_i^t$ for all $i \in \mathcal{D}$, column total entries $x_{m+1,j}^t = \alpha_j \sum_{i \in \mathcal{D}} Q_i^t$ for all $j \in \mathcal{C}$, and grand total entry $x_{m+1,n+1}^t = \sum_{i \in \mathcal{D}} Q_i^t$. Fair shares specify the fraction of seats a category is entitled to receive as per the reservation scheme until period t. The internal entry x_{ij}^t represents the **period-t fair share for category j in department i**. The **period-t fair share for a category c_j in the university** is denoted by column total entry $x_{m+1,j}^t$. The grand total entry $x_{m+1,n+1}^t$ represents the cumulative sum of vacancies at the university.

For instance, consider a problem $\Lambda^2 = (\{d_1, \ d_2\}, \{c_1, \ c_2\}, \boldsymbol{\alpha} = [0.1, 0.9], (\mathbf{q}^1, \mathbf{q}^2) = ([9, 8], [17, 7]))$. Figure 1 illustrates its period-1 and period-2 fair share tables. There are two departments $\mathcal{D} = \{d_1,\ d_2\}$, corresponding to rows in the tables, and two categories $\mathcal{C} = \{c_1,\ c_2\}$, corresponding to columns. The reservation scheme reserves 10% positions in the university for members of category c_1. In period-1, department d_1 has 9 and department d_2 has 8 positions, represented by the column 3 of X^1. In period-2, department d_1 has 17 and department d_2 has 7 positions. Therefore, period-2 cumulative sums of vacancies in departments d_1 and d_2 are 26 and 15, represented by the column 3 of X^2. The first column of table X^1 (X^2) represents the period-1 (period-2) fair shares associated with the category c_1 and the second column represents the period-1 (period-2) fair shares associated with category c_2. The first row of X^1 (X^2) represents the period-1 (period-2) fair shares associated with the department d_1 and the second row represents the period-1 (period-2) fair shares associated with department d_2.

Figure 1: FAIR SHARE TABLES

$$X^1 = \begin{array}{cc|c} 0.9 & 8.1 & 9 \\ 0.8 & 7.2 & 8 \\ \hline 1.7 & 15.3 & 17 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 2.6 & 23.4 & 26 \\ 1.5 & 13.5 & 15 \\ \hline 4.1 & 36.9 & 41 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

A two-way table is **additive** if entries add along the rows and columns to all corresponding totals. A **period-t reservation table** for the problem Λ^t is a $(m+1) \times (n+1)$ non-negative integer two-way table $\bar{X}^t = (\bar{x}^t_{ij})$, with rows indexed by $i \in \mathcal{D} \cup \{m+1\}$ and columns by $j \in \mathcal{C} \cup \{n+1\}$, such that $\bar{X}^t$ is additive and $\bar{x}^t_{i,n+1} = x^t_{i,n+1}$ for all $i \in \mathcal{D}$. The internal entry $\bar{x}^t_{ij}$ represents the **period-t reservation for category j in department i**. The **period-t reservation for a category j in the university** is denoted by column total entry $\bar{x}^t_{m+1,j}$. We denote by $\bar{\mathcal{X}}$ the set of reservation tables.

A **period-t sequence of fair share tables** for the problem Λ^t is a sequence of two-way tables $Y^t = (X^1, \dots, X^t)$, where table X^s is the period-s fair share table for all $s \in \{1, 2, \dots, t\}$. We denote by $\mathcal{Y}^t$ the set of all period-t sequences of fair share tables. Given a sequence of tables Y^t, if $Y^t = (Y^{t-1}, X^t)$, then we say that Y^t **follows** Y^{t-1}.

3.2 Deterministic Solutions and Properties

A **deterministic solution** $R : \cup_{s=1}^{T} \mathcal{Y}^s \to \bar{\mathcal{X}}$ maps each sequence of fair share tables to a reservation table such that, for any $Y^t \in \cup_{s=1}^{T} \mathcal{Y}^s$,

1. $R(Y^t)$ is a period-t reservation table, and

2. $R(Y^t) \geq R(Y^{t-1})$ for all Y^t that follow Y^{t-1}.[9]

Part 2 of definition incorporates the idea that reservations are irreversible. We denote by $\mathcal{R}^T$ the set of deterministic solutions for reservation problems of length T.

For instance, Figure 2 illustrates two possible deterministic solutions for the problem depicted in Figure 1.

Figure 2: TWO DETERMINISTIC SOLUTIONS

$$X^1 = \begin{array}{cc|c} 0.9 & 8.1 & 9 \\ 0.8 & 7.2 & 8 \\ \hline 1.7 & 15.3 & 17 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 2.6 & 23.4 & 26 \\ 1.5 & 13.5 & 15 \\ \hline 4.1 & 36.9 & 41 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$R_1(Y^1) = \begin{array}{cc|c} 1 & 8 & 9 \\ 1 & 7 & 8 \\ \hline 2 & 15 & 17 \end{array}$$

(c) PERIOD-1 RESERVATION TABLE

$$R_1(Y^2) = \begin{array}{cc|c} 3 & 23 & 26 \\ 1 & 14 & 15 \\ \hline 4 & 37 & 41 \end{array}$$

(d) PERIOD-2 RESERVATION TABLE

$$R_2(Y^1) = \begin{array}{cc|c} 0 & 9 & 9 \\ 0 & 8 & 8 \\ \hline 0 & 17 & 17 \end{array}$$

(e) PERIOD-1 RESERVATION TABLE

$$R_2(Y^2) = \begin{array}{cc|c} 3 & 23 & 26 \\ 1 & 14 & 15 \\ \hline 4 & 37 & 41 \end{array}$$

(f) PERIOD-2 RESERVATION TABLE

We denote by $R(y^t)$ and x^t the internal and totals entries of $R(Y^t)$ and X^t, respectively. The ideal solution would be the fair share table if we were allowed to reserve fractional seats. Therefore, it is natural to consider integral seat allocations with entries rounded to an adjacent integer of the fair share table entries as an ideal solution. We next formulate this idea.

[9] The relation "is greater than or equal to", denoted "$\geq$", compares tables entry-wise; that is, $X \geq X'$ if, for all $(1 \leq i \leq m+1,\ 1 \leq j \leq n+1)$, $x_{ij} \geq x'_{ij}$.

A deterministic solution R **stays within quota** if, for any Y^t,

1. R **stays within department quota**: each internal entry $R(y^t) = \lceil x^t \rceil$ or $\lfloor x^t \rfloor$, and

2. R **stays within university quota**: each total entry $R(y^t) = \lceil x^t \rceil$ or $\lfloor x^t \rfloor$.

Our property formulates the idea that a deterministic solution should not deviate from its cumulative fair share by more than one seat. In this way, everyone gets either the ceiling of its cumulative fair share or the floor of its cumulative fair share.[10] There are two dimension of staying within quota: (1) each internal entry $R(y^t_{ij})$ ($1 \leq i \leq m$, $1 \leq j \leq n$) is either x^t_{ij} rounded up or rounded down and (2) each total entry $R(y^t_{m+1,j})$ ($1 \leq j \leq n$) is either $x^t_{m+1,j}$ round up or rounded down. If a solution satisfies the former one for any problem, we say that it stays within department quota. If a solution satisfies the later one for any problem, we say that it stays within university quota. For instance, in Figure 2, the solution R_1 stays within both department and university quota; however, the solution R_2 stays within department quota only.

3.3 Lottery Solutions and Properties

Randomization is the most natural and common mechanism to use in resource allocation problems when in doubt which of two or more agents should get an indivisible object. We next introduce a function to adapt this idea.

A **lottery solution** is a probability distribution ϕ over the set of deterministic solutions, where $\phi(R)$ denotes the probability of solution R. We denote by φ^T the set of lottery solutions for reservation problems of length T.

For any sequence of fair share tables Y^t, a lottery solution ϕ induces a **period-t expected reservation table** $E_\phi(Y^t) := \sum_R \phi(R)R(Y^t)$. The internal entry (i, j) in this table represents the expected fraction of seats that category j receives at department i under ϕ.

[10] For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ are the largest integer no larger than x, i.e., floor of x, and the smallest integer no smaller than x, i.e., ceiling of x, respectively.

The column total entry $(m+1, j)$ represents the expected fraction of seats that category j receives in the university under ϕ.

Our next two properties make sure that in expectation a lottery solution always achieves the fair shares as well as in implementation it picks a reservation table that is as close as to fair shares for each departments in every period.

Definition 1. *A lottery solution ϕ is unbiased if, for any $Y^t \in \cup_{s=1}^{T} \mathcal{Y}^s$,*

$$E_\phi(Y^t) = X^t.$$

This property formulates the idea that a lottery solution should implement the fair share tables in an expected sense; that is, for any Y^t, $\sum_R \phi(R)R(Y^t) = X^t$. An unbiased lottery solution promotes ex-ante "fairness". Such solutions, on the other hand, may result in an "unfair" outcome ex-post, in which one category receives all seats, while others receive none. In other words, the ex-post outcome can differ greatly from the fair share tables. To avoid this, we next extend the staying within quota property to lottery solutions.

Definition 2. *A lottery solution ϕ **stays within quota** if, for any R such that $\phi(R) > 0$,*

1. *R stays within department quota, and*
2. *R stays within university quota.*

We study lottery solutions ϕ that only pick deterministic solutions that stay within quota. There are two dimension of staying within quota. We say that a lottery solution stays within department quota if it only gives positive probabilities to deterministic solutions that stays within department quota. We say that a lottery solution stays within university quota if it only gives positive probabilities to deterministic solutions that stays within university quota.

4 Solutions from India and their Shortcomings

There are two solutions seen in practice in India, the Government's solution and the Court's solution. Both solutions use a tool called roster to determine the number of positions to be reserved. Formally, a **roster** $\sigma : \{1, 2, \dots\} \to \mathcal{C}$ is an ordered list over the set of categories $\mathcal{C}$. A roster assigns each position a category so that for any number of total positions, the number of positions to be reserved are clearly laid out. Since only a few seats might arise every period, the objective of maintaining a roster is to ensure that, over a period of time, each category gets its affirmative action policy prescribed percentage of seats.

Maintaining rosters is central to implementation of reservations in India.[11] It makes uniform and transparent implementation of the reservation policy across various government departments possible. However, maintaining rosters for educational institutions raises additional complications. Does each department in a university maintain its own roster? Or does the university as a whole maintain a roster? These questions gave rise to two solutions in India.

Before illustrating the solutions, we first introduce an example that makes the solutions easier to comprehend. The example will also be sufficient to demonstrate the various shortcomings of the two solutions.[12]

Example 1. *Consider a problem* $\Lambda^3 = (\{d_1,\ d_2,\ d_3,\ d_4\}, \{c_1,\ c_2\}, \boldsymbol{\alpha} = [1/3, 2/3], (\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3) = ([2, 1, 2, 1], [2, 1, 2, 1], [2, 1, 2, 1]))$*. Figure 3 illustrates its period-1, period-2, and period-3 fair share tables. The reservation scheme reserves 1/3 of the positions in the university for members of category* c_1*. Each period, department* d_1*,* d_2*,* d_3*, and* d_4 *have 2, 1, 2, and 1 positions, respectively. Therefore, period-2 cumulative sums of vacancies in departments are 4, 2, 4, and 2, respectively. And, period-3 cumulative sums of vacancies in departments are 6, 3, 6,*

[11] See Figure 17 and Figure 18 for the rosters prescribed by Government of India.

[12] An example with two categories and two department is also sufficient to demonstrate the shortcomings. Example 1 is constructed so that it not only illustrates the shortcomings of the both solutions, but it also demonstrates the differences between the Court's and the Government's solutions.

and 3, respectively. The roster is

$$\sigma(k) = \begin{cases} c_1, & \textit{if } k \textit{ is a multiple of } 3 \\ c_2, & \textit{otherwise} \end{cases}$$

Figure 3: FAIR SHARE TABLES

$$X^1 = \begin{array}{cc|c} 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$X^3 = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(c) PERIOD-3 FAIR SHARE TABLE

We will see that the choice of the roster in Example 1 is not the source of the shortcomings of the Government's and Court's solutions. The source of problem is that they do not account for interdependence of the departments and the university in calculating reserved seats.

4.1 Government's Solution and its Shortcomings

The Government's solution treats the *university as the unit.* That is, positions across all departments are pooled together and the roster is maintained at the university level.

For the problem in Example 1, in period-1, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 1st and 2nd positions in the roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 3th position in the roster (i.e., $\sigma(3) = c_1$).[13] Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 4th and 5th positions in the roster (i.e., $\sigma(4) = c_2,\ \sigma(5) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 6th position in the roster (i.e., $\sigma(6) = c_1$). The period-1 reservation table is illustrated by $R_G(Y^1)$ in Figure 4.

[13]When pooling positions across departments, a fixed order over departments is required to apply to the roster. In India, the alphabetic order over departments is used.

In period-2, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 7th and 8th positions in the roster (i.e., $\sigma(7) = c_2,\ \sigma(8) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 9th positions in the roster (i.e., $\sigma(9) = c_1$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 10th and 11th positions in the roster (i.e., $\sigma(10) = c_2,\ \sigma(11) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 12th position in the roster (i.e., $\sigma(12) = c_1$). The period-2 reservation table is illustrated by $R_G(Y^2)$ in Figure 4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_G(Y^3)$ in Figure 4.

Figure 4: COURT'S AND GOVERNMENT'S SOLUTION

$$X^1 = \begin{array}{cc|c} 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ 2/3 & 4/3 & 2 \\ 1/3 & 2/3 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ 4/3 & 8/3 & 4 \\ 2/3 & 4/3 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$X^3 = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(c) PERIOD-3 FAIR SHARE TABLE

$$R_G(Y^1) = \begin{array}{cc|c} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \\ \hline 2 & 4 & 6 \end{array}$$

(d) PERIOD-1 RESERVATION TABLE

$$R_G(Y^2) = \begin{array}{cc|c} 0 & 4 & 4 \\ 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 0 & 2 \\ \hline 4 & 8 & 12 \end{array}$$

(e) PERIOD-2 RESERVATION TABLE

$$R_G(Y^3) = \begin{array}{cc|c} 0 & 6 & 6 \\ 3 & 0 & 3 \\ 0 & 6 & 6 \\ 3 & 0 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(f) PERIOD-3 RESERVATION TABLE

$$R_C(Y^1) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ \hline 0 & 6 & 6 \end{array}$$

(g) PERIOD-1 RESERVATION TABLE

$$R_C(Y^2) = \begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 2 \\ \hline 2 & 10 & 12 \end{array}$$

(h) PERIOD-2 RESERVATION TABLE

$$R_C(Y^3) = \begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ \hline 6 & 12 & 18 \end{array}$$

(i) PERIOD-3 RESERVATION TABLE

Period-3 reservation for category c_1 in department d_1 and department d_3 is 0, however, the fair share is 2 positions. Moreover, period-3 reservation for category c_1 in department d_2 and department d_4 is 3, however, the fair share is 1 position. Therefore, the Government's solution

R_G does not stay within department quota. Moreover, in Example 1, if the departments had the same number of positions for the next periods, department d_1 and department d_3 would not reserve any seats for category c_1, and department d_2 and department d_4 would not reserve any seats for category c_2.

Two shortcomings of the Government's solution R_G are revealed by Example 1:

1. The Government's solution R_G does not stay within quota.
2. The Government's solution R_G allows for large deviations in seat allocations from fair shares at the department level.

Essentially, Example 1 shows that treating university as the unit can lead to outcomes that fail to follow the reservation policy at the department level.[14]

4.2 Court's Solution and its Shortcomings

The Court's solution treats *department as the unit*. That is, positions are not pooled across departments. Instead, each department independently maintains a roster.

For the problem in Example 1, in period-1, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 1st and 2nd positions in its roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 1st position in its roster (i.e., $\sigma(1) = c_2$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 1st and 2nd positions in its roster (i.e., $\sigma(1) = c_2,\ \sigma(2) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 1st position in its roster (i.e., $\sigma(1) = c_2$). The period-1 reservation table is illustrated by $R_C(Y^1)$ in Figure 4.

In period-2, department d_1 has two positions: The number of positions reserved for department d_1 is determined by the 3th and 4th positions in its roster (i.e., $\sigma(3) = c_1,\ \sigma(4) =$

[14] In fact, in Proposition 3, we show that for any solution that stays within university quota, the deviations in seat allocations from fair shares at the department level can not be limited by a fixed number.

c_2). Department d_2 has one position: The number of positions reserved for department d_2 is determined by the 2nd positions in its roster (i.e., $\sigma(2) = c_1$). Department d_3 has two positions: The number of positions reserved for department d_3 is determined by the 3th and 4th positions in its roster (i.e., $\sigma(3) = c_1, \ \sigma(4) = c_2$). Department d_4 has one position: The number of positions reserved for department d_4 is determined by the 2nd position in its roster (i.e., $\sigma(2) = c_1$). The period-2 reservation table is illustrated by $R_C(Y^2)$ in Figure 4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_C(Y^3)$ in Figure 4.

Period-1 reservation for category c_1 in the university is 0, however, the fair share is 2 positions. Moreover, period-2 reservation for category c_1 in the university is 2, however, the fair share is 4 positions. Therefore, the Court's solution R_C does not stay within university quota. Moreover, in Example 1, if there were 4 more departments d_5, d_6, d_7, and d_8, with the same number of positions as department d_1, d_2, d_3, and d_4, respectively, period-1 reservation for category c_1 in the university would still be 0. And, period-2 reservation for category c_1 in the university would still be 4 while the fair share was 8 positions.

Two shortcomings of the Court's solution R_C are revealed by Example 1:

1. The Court's solution R_C does not stay within quota.

2. The Court's solution R_C allows for large deviations in seat allocations from fair shares at the university level.

Essentially, Example 1 shows that treating department as the unit can lead to outcomes that fail to follow the reservation policy at the university level.

5 Theoretical Results

5.1 Single Period Results

One way to approach the problem of reservation in two dimensions is to ignore the time dimension, that is, the problem can be treated as an independent problem in each period.[15] In that case, a lottery solution that is unbiased and stays within quota always exists.

Proposition 1. *There exists a lottery solution $\phi \in \varphi^1$ that is unbiased and stays within quota.*

The proof, presented in Appendix A, uses an adaptation of the Cox (1987) controlled rounding procedure to construct a unbiased lottery solution that stays within quota. By Proposition 1, any period-1 fair share table is implemented by a lottery solution that only gives positive probability to period-1 reservation tables that do not deviate from fair shares by more than one seat. The following corollary directly follows Proposition 1.

Corollary 1. *There exists a deterministic solution $R \in \mathcal{R}^1$ that stays within quota.*

Corollary 1 implies that for any problem of length $T = 1$, there always exists a reservation table that stays within quota. That is, there is a satisfactory solution to the problem of reservation in two dimensions if in each period the problem is treated independently.

5.2 Multi Period Results

Treating each period's problem independently can lead to adverse outcomes over time. In particular, since integer seat allocations differ from the fair share tables in every period, accumulation of these differences can result in large deviation from fair shares over time. We next show this issue in an example.

[15] This is analogous to biproportional apportionment problems. In some proportional electoral systems with more than one constituency the number of seats must be allocated to parties within territorial constituencies, as well as, the number of seats that each party has to receive at a national level.

Example 2. *Consider a problem depicted in the following fair share table, with two departments d_1, d_2 having 2 and 7 positions, respectively, and two categories c_1, c_2, and the reservation scheme vector $\boldsymbol{\alpha} = [0.1, 0.9]$.*

The following deterministic solution stays within quota, but it does not give any positions to category c_1.

$$X = \begin{array}{cc|c} 0.2 & 1.8 & 2 \\ 0.7 & 6.3 & 7 \\ \hline 0.9 & 8.1 & 9 \end{array}$$

(a) FAIR SHARE TABLE

$$R(X) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 7 & 7 \\ \hline 0 & 9 & 9 \end{array}$$

(b) RESERVATION TABLE

Example 2 suggests that not reserving any seats is a solution that stays within quota. For instance, a university can repeatedly apply this solution to each period's problem and does not reserve a single seat.

In general case, a lottery solution ϕ that treats each period's problem independently rounds up or rounds down each fair share with some probabilities. Therefore, in the range of the lottery solution ϕ, there exists an outcome that rounds down a particular entry in every period. That is, the lottery solution ϕ can result in seat allocations with sizeable deviations from fair shares.

We next examine how our single period results extend to the multi-period problem. We first show that for every problem, a deterministic solution that stays within quota does not always exists.

Proposition 2. *There does not exist a deterministic solution $R \in \mathcal{R}^T$ that stays within quota for $T > 1$.*

Proposition 2 implies that for every problem of length $T > 1$, unlike single period, solutions deviate from fair shares by more than one seat. It also implies that it is impossible to both stay within university quota and stay within department quota. We next generalize staying within quota property to allow for some differences in fair shares and seat allocations.

A **bias of a deterministic solution** R at Y^t is a two-way table $bias(R(Y^t))$, with each entry $bias(R(y^t)) := R(y^t) - x^t$. The bias of a solution is the difference between the solution and the fair share table. With this definition, a solution stays within quota if, for any Y^t, each entry $|\text{bias}(R(y^t))| < 1$, that is, for any problem, the bias of the solution is always less than 1 in absolute value. Our next property allows a solution to deviate from fair shares up to a constant number.

Definition 3. *A deterministic solution $R \in \mathcal{R}^T$ has a **finite bias** if there exists a constant $b > 0$ such that, for any $Y^t \in \cup_{s=1}^T \mathcal{Y}^s$,*

$$|bias(R(y^t))| < b.$$

One might be tempted to think that there would be solutions that allow for larger deviations in seat allocations from fair shares at the department level but stays within university quota. We show that such solutions do not exist.

Proposition 3. *There does not exist a deterministic solution $R \in \mathcal{R}^T$ that has a finite bias and stays within university quota for $T > 1$.*

The proof is in Appendix A. Proposition 2 is a corollary of Proposition 3. By Proposition 2 we learn that any procedure that stays within university quota cannot stay within department quota. By Proposition 3 we learn that any procedure that stays within university quota can lead to departments to grow in size over time without reserving a single seat.

Proposition 2 and Proposition 3 have a stronger implication: there is no deterministic solution to the problem of reservation in two dimensions that stays within quota. This negative result provides yet another reason to use lottery solutions to address the problem of reservation in two dimensions.

We next present the main existence result: the set of lottery solutions that are unbiased and stay within department quota is non-empty.

Theorem 1. *There exists a lottery solution ϕ that is unbiased and stays within department quota.*

A formal proof of Theorem 1 is presented in Appendix A. The proof utilizes a network flow to construct a lottery over rosters. Each department is then assigned a roster drawn independently from the constructed lottery. This two-step procedure induces a lottery solution, denoted ϕ^* and defined formally in Appendix A. The lottery solution is shown to be unbiased and stays within department quota, that is, each category gets (i) ex-ante its fair share, and (ii) ex-post its fair share either rounded up or down in every department.[16]

Theorem 1 implies that there is a lottery solution that ensures that each department sticks to the reservation scheme while the university, as a whole, respects the fair shares in an expected sense. By Proposition 3, however, we know that such solutions can result in biases greater than one at the university level. To show that our lottery solution limits the probability of these occurrences, we modify the staying within university quota property.

We denote the outcome of a lottery solution ϕ at a sequence of fair share tables Y^t by the random variable Z^t and its entries by z^t_{ij}. The deviation of the outcome of lottery solution ϕ for a category $j \in \mathcal{C}$ in the university is $z^t_{m+1,j} - x^t_{m+1,j}$. This random variable measures the deviation of the seat allocation at the university level from its fair share.

Definition 4. *A lottery solution ϕ* ***approximately stays within university quota*** *if, for any Y^t, for any category $j \in \mathcal{C}$ and for any $b > 0$, we have*

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) \leq e^{-\frac{b^2}{3x^t_{m+1,j}}} \ ,$$

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) \leq e^{-\frac{b^2}{2x^t_{m+1,j}}} \ .$$

We establish probabilistic concentration bounds on the deviations for our lottery solution

[16] One can show that the set of lottery solutions that are unbiased and stay within university quota is also non-empty. However, staying within the department quota property better suits our applications because one goal of affirmative policies is to increase diversity in all sub-units (departments and university as a whole), and the smallest sub-units in our setup are departments.

ϕ^* and show that ϕ^* approximately stays within university quota.

Theorem 2. *The lottery solution ϕ^* is unbiased, stays within department quota, and approximately stays within university quota.*

Theorem 2 follows from a Chernoff-type concentration bound. We establish the probability bounds in a fashion similar to Gandhi et al. (2006). By this property, the probability of deviating from university quota by a value greater than b decays exponentially with b^2. Therefore, there is a procedure that ensures that each department obeys the reservation scheme, while the university as a whole approximately follows the reservation scheme.

We next show that the bounds in Definition 4 are tight (up to a multiplicative constant in the exponent) and thus rules out any improvement of the deviation of the seat allocation at the university level from its fair share.

Proposition 4. *Consider a lottery solution that is unbiased, stays within department quota and limits the probability of deviation of the seat allocation at university level in following way: for any Y^t, for any category $j \in \mathcal{C}$ and for any $b > 0$, the lottery satisfies*

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) \leq f(x^t_{m+1,j}, b) \ ,$$

$$Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) \leq f(x^t_{m+1,j}, b) \ .$$

Then, there exists a constant $k > 0$ such that for any $b > 0$,

$$\lim_{x^t_{m+1,j} \to \infty} \frac{e^{-\frac{b^2}{x^t_{m+1,j}}k}}{f(x^t_{m+1,j}, b)} = 0 \ .$$

Proposition 4 shows that there exists a constant $k > 0$ such that any lottery that is unbiased and stays within department quota can approximately stay within university quota (in the sense of Definition 4) with a probabilistic guarantee no better than $e^{-\frac{b^2}{x^t_{m+1,j}}k}$. A proof of Proposition 4 is presented in Appendix A.

6 Empirical Study of Reservation in Two Dimensions

Here we present a comprehensive evaluation of recruitment advertisements to highlight the severity of shortcomings in the existing solutions and to reflect the benefits of adopting our proposed solutions. Specifically, we evaluate the general quality of the advertised two-way apportionments with respect to the instances and magnitude of quota violations, and present the advantage our proposed solution exhibits.

Our data comprises 60 advertisements released in the following five recruitment settings where two-dimensional reservation problems are seen in practice.

1. Assistant Professors of University of Delhi
2. Officers of Indian Administrative Services
3. Officers of Indian Forest Services
4. Officers of Indian Police Services
5. Assistants of Reserve Bank of India

In the preceding sections we presented and analyzed the problem in the context of a university. Therefore, we will continue to use the same terminology for all advertisements. The term *departments* refers to departments in a university for the assistant professors advertisements. However, for other advertisements the departments correspond to the states (in India) where an officer or an assistant shall be recruited. Similarly, the term *university* corresponds to the country (India) in the latter advertisements.

An overview of the recruitment advertisement data is presented in Table 2. The advertisements provide a variety of two-dimensional reservation problems with the number of departments varying from 8 to 50; the number of vacancies in a department varying from 1 to 30; and the number of vacancies in the university varying from 21 to 1000. The advantage of using data from different institutions is that the variety of procedures used at these institutions help highlight the robustness of shortcomings we discussed in Section 4.

Table 2: OVERVIEW OF RECRUITMENT ADVERTISEMENTS

		Departments		Dept. Vacancies		Total Vacancies	
Institution	Ads	Avg.	Min-Max	Avg.	Min-Max	Avg.	Min-Max
University of Delhi	23	19.7	8-50	4.2	1-10	94.8	21-405
Indian Administrative Services	15	24.7	24-26	9.47	5-15	148.9	87-180
Indian Forest Services	7	25.1	24-26	3.6	3-4	95.4	78-110
Indian Police Services	8	25.3	24-26	12.8	10-16	150.1	148-153
Reserve Bank of India	7	17	17-17	23.7	13-30	648.1	500-1000

6.1 Single Period Analysis

First consider the problem of reservations as a single period problem. Thus in this subsection each advertisement is treated as an independent single period two-dimensional reservations problem. In line with our theoretical analysis, we use the department and university quota violations in judging the quality of solutions advertised.

Table 3 shows that the instances of both the department quota and the university quota violations are pervasive in the advertised solutions of all the institutions. The percentage of instances of violations, obtained by dividing the number of violations that occurred by the maximum number of violations possible, is an informative summary measure. Based on this measure, the probability that a typical category would witness a department quota violation in a typical department ranges from 0.08 in University of Delhi to 0.59 in Reserve Bank of India. The probability that a typical category would witness a university quota violation ranges from 0.18 in India Forest Services to 0.93 in Reserve Bank of India.

In order to provide a complete picture of the severity of shortcomings, we present the magnitude of bias (in cases of quota violation) in Table 3. The magnitude of bias is the absolute value of bias as defined in Section 5.2. At the department level, this measure shows that, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.3 in University of Delhi to 4.1 in Reserve Bank of India. At the university level, this measure shows that, in case of quota violation, the average deviation from fair

Table 3: SINGLE PERIOD QUOTA VIOLATIONS – STATISTICS

	Instances of Violations				Magnitude of Bias	
	Avg.	Min-Max	Total	Percentage	Avg.	Min-Max
University of Delhi						
Department Quota	6.8	0-24	156	8%	1.3	1-4
University Quota	2.6	1-5	60	59.4%	2.7	1-13
Indian Administrative Services						
Department Quota	28.9	2-48	434	29.2%	1.8	1-6.5
University Quota	1.9	0-4	28	46.7%	3.9	1-6.9
Indian Forest Services						
Department Quota	17.6	8-24	123	17.5%	1.5	1-2.9
University Quota	0.7	0-2	5	17.9%	1.4	1.3-1.5
Indian Police Services						
Department Quota	32.8	27-38	262	32.5%	1.8	1-5.1
University Quota	2.9	1-4	23	71.9%	2.3	1.2-5.3
Reserve Bank of India						
Department Quota	40.1	34-49	281	59%	4.1	1-35.8
University Quota	3.7	3-4	26	92.9%	20.8	2.5-60.6

shares for a typical category ranges from 1.4 in Indian Forest Services to 20.8 in Reserve Bank of India.

As a single period problem, the two-dimensional reservations problem has been shown to admit an elegant solution called *controlled rounding* that stays within quota (see section Section 5.1). If each reservation problem were to be treated independently, adopting controlled rounding procedure for making reservation tables would lead no quota violations. Therefore making it possible to achieve simultaneously the prescribed percentage of reservations at both the department and the university level in single period problems (as shown in Proposition 1).

6.2 Multi Period Analysis

In Section 2, with emphasis on maintaining rosters, the intent of India's policymakers is clear. In the face of the indivisibility of seats, their policies aim to achieve the prescribed percentage of reservations not in a single period but over time. Therefore, analysis of the recruitment data is incomplete without checking whether the quota and biases cancel out and consequently disappear over time. For this purpose we need to look at sequences of consecutive advertisements that share the same set of departments and the same reservation policy. There are seven such sequences in our data.

Results from the last period of these seven sequences of consecutive advertisements in Table 4 show that the single period violations are not cancelling over time, rather they are adding up. Both the instances of violations and the magnitude of bias are now higher than the numbers reported in Table 3 for single period problems. The probability that a typical category would witness a department quota violation in a typical department ranges from 0.36 in Indian Forest Services to 0.88 in Reserve Bank of India. The probability that a typical category would witness a university quota violation ranges from 0.50 in Indian Forest Services to 1 in Reserve Bank of India. At the department level, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.8 in Indian Forest Services to 11.7 in Reserve Bank of India. At the university level, in case of quota violation, the average deviation from fair shares for a typical category ranges from 1.5 in Indian Forest Services to 83.2 in Reserve Bank of India.

The findings suggest that the problem worsens with time in that there are more instances of violations and larger deviations from policy prescribed percentage of reservations. This is not surprising given the negative results presented in Proposition 2 and Proposition 3. However, the scope of improvement is clear. Theorem 1 and Theorem 2 show that there exists an unbiased solution that stays within quota at the department level and approximately stays within quota at the university level. A comparison of this proposed solution with the existing solution is the point of our next simulation exercise. For this exercise we will consider the

Table 4: MULTI PERIOD QUOTA VIOLATIONS – STATISTICS

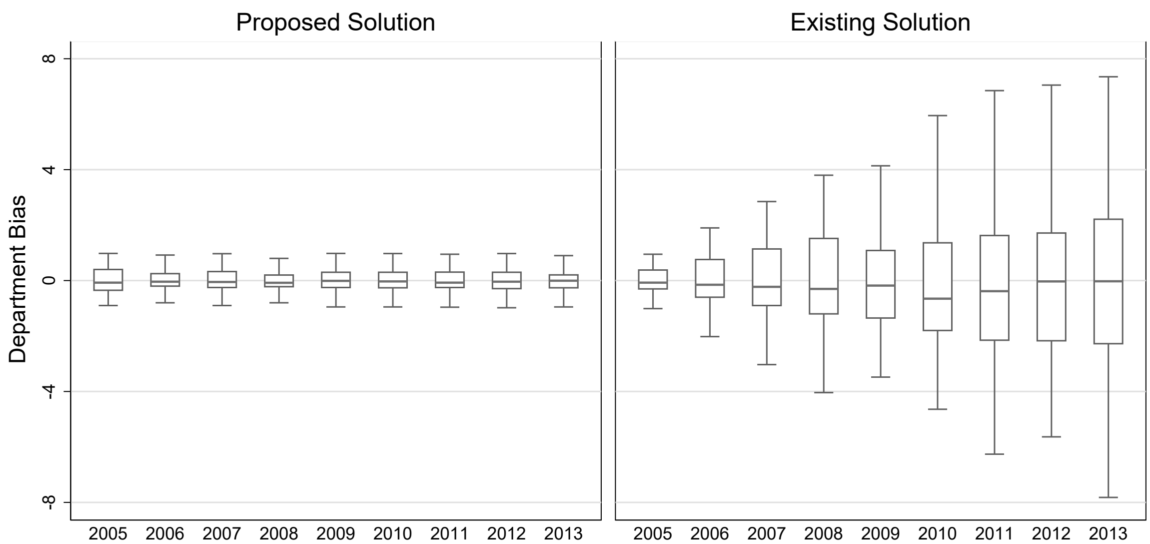
	Instances of Violations		Magnitude of Bias	
	Total	Percentage	Avg.	Min-Max
Indian Administrative Services: 2005 to 2013				
Department Quota	76	79.2%	3.5	1-14.1
University Quota	4	100%	16.6	5.2-28
Indian Administrative Services: 2014 to 2018				
Department Quota	79	75.9%	4.1	1-18.1
University Quota	2	50%	4	3.5-4.4
Indian Forest Services: 2011 to 2013				
Department Quota	35	36.5%	1.8	1-3.1
University Quota	2	50%	1.5	1.2-1.8
Indian Forest Services: 2015 to 2018				
Department Quota	54	51.9%	2.8	1.1-6.9
University Quota	3	75%	3.1	2-4.9
Indian Police Services: 2010 to 2011				
Department Quota	45	46.9%	2.2	1-4.2
University Quota	4	100%	2.6	1.7-3.5
Indian Police Services: 2014 to 2018				
Department Quota	73	70.2%	3.4	1.1-12.6
University Quota	3	75%	5	2-7.7
Reserve Bank of India: 2012 to 2017				
Department Quota	60	88.2%	11.7	1-35.5
University Quota	4	100%	83.2	10-166.4

longest sequence of consecutive advertisements in our data: the advertisement of Indian Administrative Services from 2005 to 2013.

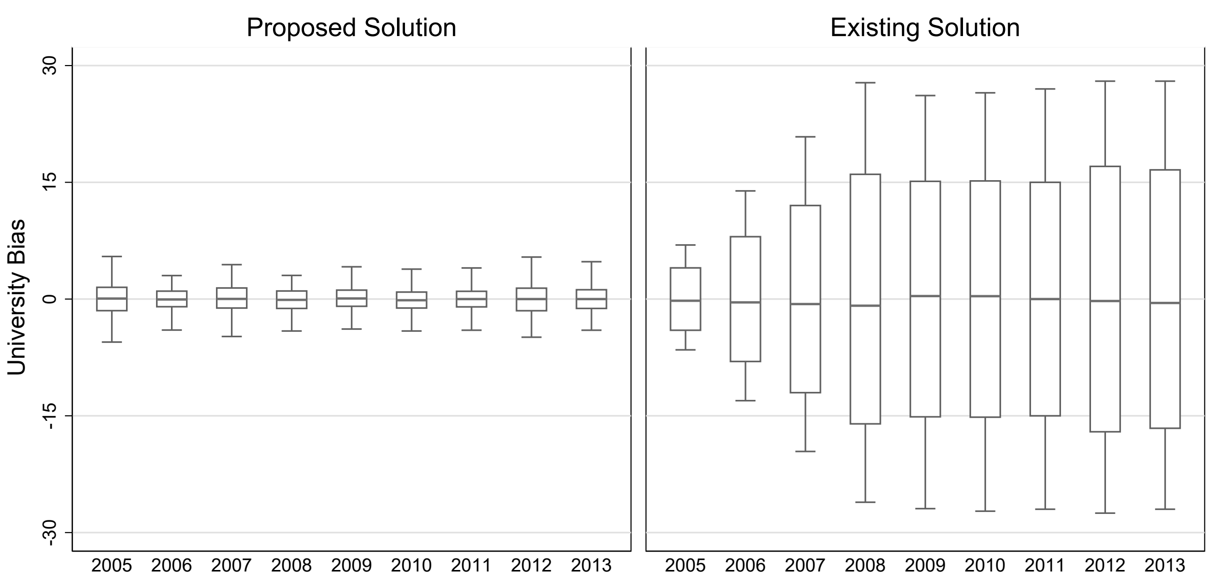
The objective of the simulation exercise is to compare the evolution of bias over time under the existing solution with the solution proposed in this paper. For this purpose, we simulate a set of 50 advertisements adhering to the proposed solution and plot the bias at each time period in Figure 6. The top-left panel shows that, for the proposed solution's advertisements, the department bias stays well within the $[-1, 1]$ interval, that is, there are no quota violations at the department level. In contrast, under the existing (advertised) solution presented in the top-right panel, the bias accumulates over time at the department level. The bottom-left panel shows that though the university violations occur under the proposed solution, the bias does not add up over time. The significance is apparent when one compares it to the evolution of bias under the existing solution presented in the bottom-right panel.

Figure 6: BIASES OF PROPOSED AND EXISTING SOLUTIONS

(a) DEPARTMENT BIAS OVER TIME



(b) UNIVERSITY BIAS OVER TIME



Note: Box plots show medians, quartiles, and adjacent values of bias distributions over time.

7 Conclusion

This paper has offered an analysis of two-dimensional reservation problems using the theory of apportionment and rounding problems. We have theoretically and empirically documented the shortcomings of existing solutions and proposed a solution with demonstrable advantages over the existing solutions. From a broader perspective, even though our search for quality solutions is limited to the *staying within quota* property, the analysis here can be viewed as illustrative of the substantial scope for improvement in existing procedures for two-dimensional reservation problems.

Two-dimensional reservation problems are open to several alternative approaches that deserve extra work. A particular one that deserves mention is the error minimization approach that has yielded a class of methods to solve biproportional apportionment problems (Ricca et al. (2012), and Serafini and Simeone (2012)). These methods take a fractional matrix as the target (fair share table in our case) and solve a constrained optimization problem where the objective corresponds to a measure of the error between the solution and the target matrix. Such an approach may pave the way to a richer study of defining and finding appealing solutions to two-dimensional reservation problems.

Our problem also suggests possible extensions in the theory of apportionment. The multi-period considerations introduced in this paper could be worth exploring in the classic biproportional apportionment problem context of translating electoral votes into parliamentary seats.

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Appendices for Online Publication

A Proofs

Proof of Theorem 1

The proof is constructive and has two parts. We first define the Roster-Finding Algorithm, which takes a reservation scheme vector as inputs and generates a random roster as an output, that is, a lottery over rosters.We then assign the random roster to each department independently. The random roster is constructed such that if every department follows it, the induced solution stays within the department quota. We denote this solution as our lottery solution ϕ^*. We, lastly, show that the lottery solution ϕ^* is unbiased.

Proof of Theorem 1. Let $\mathcal{C}$ be the set of categories and $\boldsymbol{\alpha} = [\alpha_j]_{j\in\mathcal{C}}$ be the reservation scheme. Let P represent the given reservation scheme as a $k \times n$ two-way table, where the rows denote the index of the seats and the columns denote the categories. The internal entry p_{ij} equals to α_j for every (i, j). Let assume that for each column, entries sum up to an integer (if there is a common multiplier for fractions in the reservation scheme vector, then such k exists).[17] The output of the algorithm will be an integral table that define how a department reserves its positions over time, i.e., a roster. We next construct a set of constraints that bounds the elements of the table P.

For each constraints K, let $\underline{p_K}$ and $\bar{p_K}$ be the floor and ceiling of the constraint. That is, $\underline{p_K} = \lfloor \sum_{(i,j)\in K} p_{ij} \rfloor$ and $\bar{p_K} = \lceil \sum_{(i,j)\in K} p_{ij} \rceil$. We will consider tables P' that satisfying, for each K,

$$\underline{p_K} \leq \sum_{(i,j)\in K} p'_{ij} \leq \bar{p_K}.$$

We have three types of constraints. Internal constraints make sure that each internal

[17]The generalization to non-integer sums is made by constructing an extended table P' in a way that is equivalent to P except the last row. The last row of P' is generated by taking 1- fractional part of the column totals (similar to how the extended table is created in the algorithm given for proof of Proposition 1).

entry can be either 1 or 0. Row sums are required to be one since every position is assigned to exactly one category. Column constraints make sure that difference between cumulative some of positions given to a category and cumulative fair shares is less than one.

Let $\mathcal{K}_I$ be the internal constraints, i.e., $0 \leq p'_{ij} \leq 1$ for every (i,j). Let $k_{ij} := \{(i,j)\}$ denote such constraint. Let $\mathcal{K}_R$ be the set of row constraints, i.e., $\sum_{j\in\mathcal{C}} p'_{ij} = 1$ for every i. Let $R_i := \{(i,j)|j \in \mathcal{C}\}$ denote such constraint. Let $\mathcal{K}_C$ be the set of column constraints, i.e., $\lfloor \sum_{i\leq l} p_{ij} \rfloor \leq \sum_{i\leq l} p'_{ij} \leq \lceil \sum_{i\leq l} p_{ij} \rceil$ for every $2 \leq l \leq n$ and $j \in \mathcal{C}$. Let $C_{lj} := \{(i,j)|i \leq l\}$ denote such constraint.

We next create a flow network. The set of vertices consists of the source, the sink, vertices for each $k \in \mathcal{K}_I$, each $R \in \mathcal{K}_R$, and for each $C \in \mathcal{K}_C$. The following rule governs the placement of directed edges:

1. A directed edge from source C_{nj} for every $j \in \mathcal{C}$.

2. A directed edge from C_{lj} to k_{lj} and C_{l-1j} for every $l \geq 3$ and $j \in \mathcal{C}$.

3. A directed edge from C_{2j} to k_{2j} and k_{1j} for every $j \in \mathcal{C}$.

4. A directed edge from k_{ij} to R_i for every (i,j).

5. A directed edge from R_i to sink for every i.

Note that the constraint structure for $\mathcal{K}_C \cup \mathcal{K}_I$ and $\mathcal{K}_R \cup \mathcal{K}_I$ are hierarchical. A set of constraints $\mathcal{K}$ is **hierarchical** if, for every pair of constraints K' and K'', we have that $K' \subset K''$ or $K'' \subset K'$ or $K' \cap K'' = \emptyset$.

We next associate flow with each edge. Notice that there is only one incoming edge for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$. And, there is only one outgoing edge for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. Observe that it is because of the hierarchical sets of constraints. Therefore, it is sufficient to associate incoming flows for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$ and outgoing flows for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. For each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$, the incoming flow is equal to $\sum_{(i,j)\in K} p_{ij}$. For each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$, the outgoing flow is equal to $\sum_{(i,j)\in K} p_{ij}$. Furthermore, the flow association ensures that the amount of incoming flow is equal to the amount of outgoing flow for each vertex.

Notice that we map table P with the constraint structures to a flow network. In addition, the mapping is injective. As long as the constraints are still satisfied after the transformation, every transformation in the flow network can be mapped back to table P.

Definition 5. *We call the pair of tables (P^1, P^2) a **decomposition** of table P, if*

1. *there exists $\beta \in (0, 1)$ such that $P = \beta P^1 + (1 - \beta)P^2$,*
2. *for each constraint K, $\underline{p_K} \leq \sum_{(i,j)\in K} p^l_{ij} \leq \bar{p_K}$ for $l = 1, 2$, and*
3. *table P^1 and P^2 have more number of integral entries than table P.*

The following constructive algorithm has two parts. We first find a cycle of fractional edges in the network flow. We then alter the flow of edges in two different ways until one edge becomes integral. It will provide us a decomposition of table P.

Roster-Finding Algorithm

Repeat the following as long as the flow network contains a fractional edge:

Step 1: Choose any edge that has fractional flow. Since the total inflow equals to total outflow for each vertex, there will an adjacent edge that has fractional flow. Continue to add new edges with fractional flows until a cycle is formed.

Step 2: Modify the flows in the cycle in two ways to create P^1 and P^2:

1. First way: the flow of each forward edge is increased and the flow of each backward edge is decreased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as d_-. Map back the resulting flow network to a two way table. Denote the table as P^1.
2. Second way: the flow of each forward edge is decreased and the flow of each backward edge is increased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as d_+. Map back the resulting flow network to a two way table. Denote the table as P^2.

3. Set $\beta = \frac{d_-}{d_- + d_+}$.
4. The pair of tables (P^1, P^2) is a decomposition of table P, where $P = \beta P^1 + (1-\beta)P^2$.

The algorithm creates a lottery over integral two-way tables that share the same constraint structure as table P.[18] Assume that $\bar{P}$ is an integral table constructed by the algorithm, and its compound probability is γ. We construct a roster by each of these integral tables as follows. For each internal entry of table $\bar{P}$, if $\bar{p}_{ij} = 1$ then assign $\sigma(i) = c_j$. We next assign probability γ to roster σ. Thus, we obtain a random roster.

Notice that the expected number seats for each category j in the first q seats equals to $q\alpha_j$ for $q = 1, 2, \ldots$. We next create the induced lottery solution ϕ^* for the problem of reservation in two dimensions as follows. We assign the random roster to each department. Each department then reserves positions according to the roster realized from the lottery. For example, if roster σ is realized for department i then, the number of positions reserved in department i in period-1 is determined by $\sigma(1), \ldots, \sigma(q_i^1)$. The number of positions reserved in department i in period 2 is determined by $\sigma(q_i^1 + 1), \ldots, \sigma(q_i^1 + q_i^2)$.

We next show that the lottery solution ϕ^* is unbiased. Given the lottery solution ϕ^* and a a sequence of fair share tables $Y^t = (X^1, \ldots, X^t)$, we denote the outcome of the lottery solution by the random variable Z^t. We know that the expected number of positions reserved to category j in department i until period-t is $E(z_{ij}^t) = \sum_{s \leq t} q_i^s \alpha_j$. Moreover, the internal entry x_{ij}^t of fair share table X^t also equals to $\sum_{s \leq t} q_i^s \alpha_j$. Thus, the lottery solution ϕ^* is unbiased.

This proves the theorem. ∎

An example for Theorem 1

To make the Roster-Finding Algorithm easier to understand and show the whole procedure that constructs the lottery solution ϕ^*, we show an example.

[18] Moreover, the expected table equals to table P.

Consider a university where there are two categories $\mathcal{C} = \{c_1, c_2\}$ and the reservation scheme is $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (1/3, 2/3)$. Suppose we wish to implement the reservation scheme in a problem of reservation in two dimensions. We represent the given reservation scheme as a two-way table P, where the rows denote the index of the positions and the columns denote the categories. Each internal entry $p_{ij} = \alpha_j$. The output of the algorithm will be an integral table that define how a department reserves its positions over time, i.e., a roster.

There are three positions for easy illustration.[19] However, this method works for more general (total $3k$ positions, where $k = 1, 2, \dots$) cases. The example table P is

$$P = \begin{array}{cc|c} 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ \hline 1 & 2 & 3 \end{array}$$

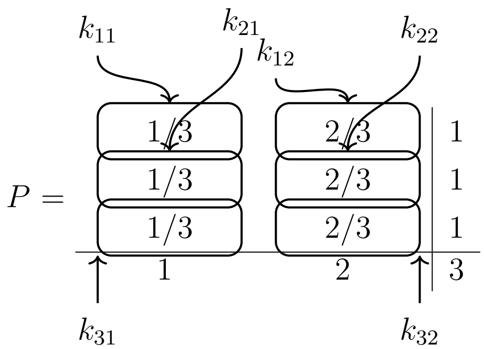
Figure 7 illustrates the constraint structure. Column constraints are $C_{31} = \{k_{11}, k_{21}, k_{31}\}$, $C_{21} = \{k_{11}, k_{21}\}$, $C_{32} = \{k_{12}, k_{22}, k_{32}\}$, and $C_{22} = \{k_{12}, k_{22}\}$, and row constraints are $R_1 = \{k_{11}, k_{12}\}$, $R_2 = \{k_{21}, k_{22}\}$, and $R_3 = \{k_{31}, k_{32}\}$.

The two-way table P with the constraints is then represented as a network flow. Starting from the source, the flows first pass through the sets in column constraints, which are arranged in descending order of set-inclusion. That is, for example, $C_{31} \supset C_{21} \supset k_{11}$. This explains the flow network on the left side of Figure 8, where the numbers on the edges represent the flows. The flows then proceed along the directed edges that represent the set-inclusion tree, eventually reaching the singleton sets. That is, for example, $k_{11} \subset R_1$. This explains the flow network on the right side of Figure 8.

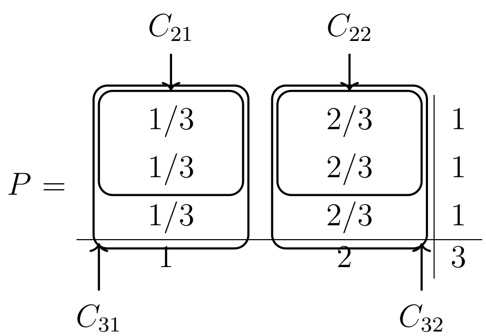
In the flow network, note that the flow associated with each edge reflects the totals of elements in the corresponding set. And, the flow arriving at each vertex equals the flow leaving that vertex. Now we are ready to present the algorithm. The algorithm will conserve these two properties while constructively find new flow network with fewer fractional

[19]Using the common multiple of the fractions in the reservation scheme, three in our case, also helps to understand.

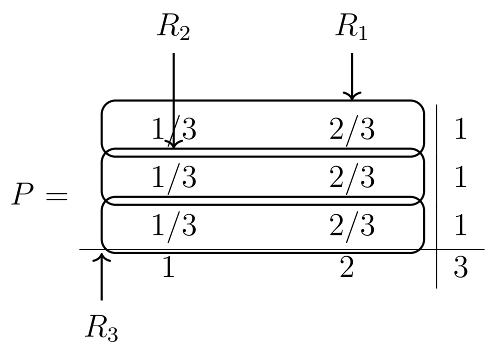
Figure 7: CONSTRAINT STRUCTURE OF THE EXAMPLE P



(a) INTERNAL CONSTRAINTS



(b) COLUMN CONSTRAINTS



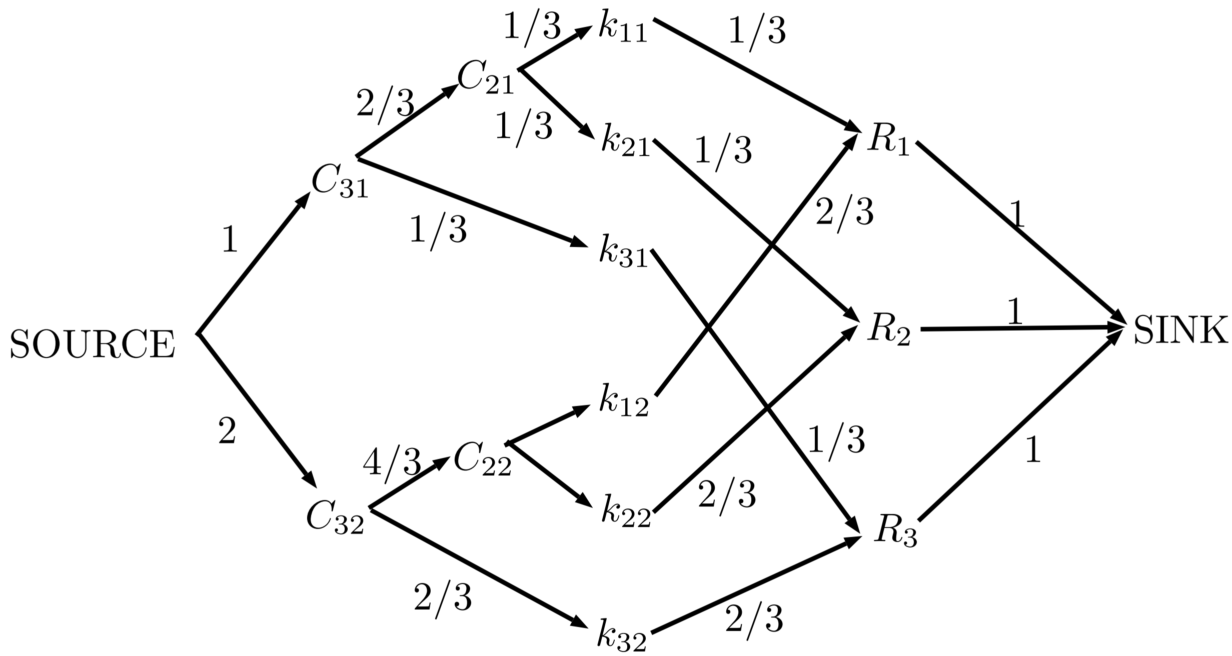
(c) ROW CONSTRAINTS

elements.

We first identify a cycle of edges with fractional flows. Choosing any fractional edge, say (C_{31}, k_{31}), we find another fractional edge that is neighbor to k_{31}. If a vertex has a fractional edge then it has to have another fractional edge: since total inflow equals to outflow for every vertices(except source and sink), we would have a contradiction. We continue to add new fractional edges until we form a cycle. In our example, the cycle of fractional edges is $C_{31} \rightarrow^{1/3} k_{31} \rightarrow^{1/3} R_3 \leftarrow^{2/3} k_{32} \leftarrow^{2/3} C_{32} \rightarrow^{4/3} C_{22} \ldots \leftarrow^{2/3} C_{31}$. We illustrates this cycle in Figure 9 with dashed lines.

Next, we alter the cycle's edge flows. We first increase the flow of each forward edge while decreasing the flow of each backward edge at the same time until at least one flow reaches an integer value. A table P_1 is created as a result of the resulting network flow. In the example, flows along all forward edges increase from 2/3 to 1, 1/3 to 2/3, and 4/3 to 5/3,

Figure 8: FLOW NETWORK REPRESENTATION OF THE EXAMPLE P



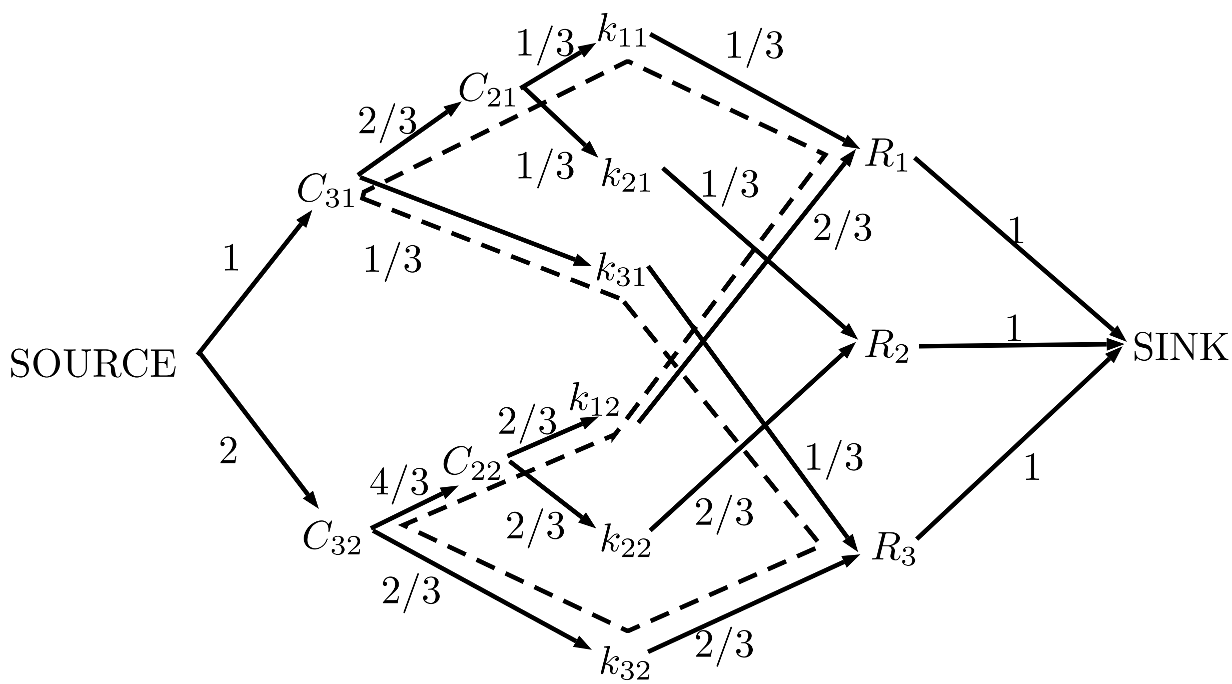
while flows along all backward edges decrease from 1/3 to 0 and 2/3 to 1/3. The adjustment is $d_+ = 1/3$. Next, the flows of the edges in the cycle are readjusted in the opposite direction, increasing those with backward edges and lowering those with forward edges in an analogous way, resulting in a new table P_2. In the example, flows along all forward edges decrease from 2/3 to 1/3, 1/3 to 0, and 4/3 to 1, while flows along all backward edges increase from 1/3 to 2/3 and 2/3 to 1. The adjustment is $d_- = 1/3$.

Now, we can decompose P into these two tables, i.e., $P = \frac{d_-}{d_- + d_+} P_1 + \frac{d_+}{d_- + d_+} P_2 = \frac{1}{2} P_1 + \frac{1}{2} P_2$. The algorithm picks P_1 with probability 0.5 and P_2 with probability 0.5. We reiterate the decomposition process until no fractions left.

At each iteration, at least one fraction in P is converted to an integer, while all current integers remain constant. Each fraction must appear in at least one iteration. As a result, the process must converge to an integer table in less iterations than the initial number of fractions in table P.

Since only the fractions along one cycle in the flow network are modified in each iteration, the expected change at this iteration for entries not on this cycle is 0, i.e., the expected

Figure 9: AN EXAMPLE OF CYCLE WITH FRACTIONAL EDGES



change in corresponding entries in P is 0. For those fractional edges that are modified, the probabilities are picked so that the expected adjustment in each iteration is 0.

Fractional edges that are adjusted multiple times will have a variety of intermediate adjustment probabilities, but because our procedure keeps the expected change at 0 in each iteration, the compound probabilities will also keep the expected change at 0.

Proof of Theorem 2

Here, we prove that lottery solution ϕ^* in Theorem 1 approximately stays university quota. In words, the lottery solution ϕ^* is designed in such a way such that it hardly ever round up (or round down) most of the entries in each column of X^t. We show the approximately staying university quota property by proving two lemmas. We first show that entries of each column of Z^t are "independent". We next prove the approximately staying university quota by applying Chernoff concentration bounds.

Lemma 1. *For any subset of $S \subset \{1, 2, \dots, m\}$ and any $j \in \{1, 2, \dots, n\}$, we have*

$$Pr\Big[\bigwedge_{i \in S} z_{ij}^t = \lceil x_{ij}^t \rceil\Big] = \prod_{i \in S} Pr\Big[z_{ij}^t = \lceil x_{ij}^t \rceil\Big] \ ,$$

$$Pr\Big[\bigwedge_{i \in S} z_{ij}^t = \lfloor x_{ij}^t \rfloor\Big] = \prod_{i \in S} Pr\Big[z_{ij}^t = \lfloor x_{ij}^t \rfloor\Big] \ .$$

Proof. Notice that the random roster is assigned to each department independently. Consequently, for any pair (i, i'), random variables z_{ij}^t and $z_{i'j}^t$ become independent, which proves the lemma. ■

Lemma 2. *For any subset of $S \subset \{1, 2, \dots, m\}$ and any $j \in \{1, 2, \dots, n\}$ with $\sum_{i \in S} x_{ij}^t = \mu$, and for any $\epsilon > 0$, we have*

$$Pr\Big[\sum_{i \in S} z_{ij}^t - \mu > \epsilon\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{3}} \ ,$$

$$Pr\Big[\sum_{i \in S} z_{ij}^t - \mu < -\epsilon\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{2}} \ .$$

Proof. We begin by recalling a result of Chernoff et al. (1952), which demonstrates that the independence property has the following large deviations result. Chernoff bounds are well-known concentration inequalities that limit the deviation of a weighted sum of Bernoulli random variables from their mean. We now use the multiplicative form of Chernoff concentration bound.

Theorem 3. *Chernoff bound: Let $A_1, A_2, \dots, A_m$ be m independent random variables taking values in $\{0, 1\}$. Let $\mu = \sum_{i=1}^m E[A_i]$. Then, for any $\epsilon \geq 0$,*

$$Pr\Big[\sum_{i=1}^m A_i \geq (1+\epsilon)\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{3}} \ ,$$

$$Pr\Big[\sum_{i=1}^{m} A_i \leq (1-\epsilon)\mu\Big] \leq e^{-\mu\frac{\epsilon^2}{2}} \ .$$

The random variable z_{ij}^t can take two values, either $\lceil x_{ij}^t \rceil$ or $\lfloor x_{ij}^t \rfloor$. If we subtract the fix number $\lfloor x_{ij}^t \rfloor$ from z_{ij}^t, then we obtain a Bernoulli distribution. Lemma 1 says that the set of random variables in each column of Z^t are independent, which means Chernoff concentration bounds hold for each column of Z^t. ■

Proof of Theorem 2. We can now prove Theorem 2. In Lemma 2, if we choose $S = \{1, \dots, m\}$, then $\sum_{i=1}^{m} x_{ij}^t = x_{m+1,j}^t$. This fact along with Lemma 2 yields our result for Theorem 2. ■

Proof of Proposition 4

Proof. For any $x_{m+1,j}^t := \mu > 0$ and any constant $b := \epsilon\mu$, we construct a problem instance. For the rest of the proof we fix category j, μ, and ϵ. This instance contains n departments, m categories. The vacancies are as follows: $q_i^s = 0$ vacancies for all $s < t$ and $q_i^t = 1$ for any $i \in \mathcal{D}$. Choose a constant $\underline{\epsilon}, \bar{\epsilon} \in (0,1)$ such that $\epsilon \in (\underline{\epsilon}, \bar{\epsilon})$. Choose $\alpha \in (0, 1/(1+\bar{\epsilon}))$ such that μ/α is an integer. Let $m = \mu/\alpha$. For category j, α fraction of vacancies are to be reserved. Note that, by definition, $x_{ij}^t = \alpha$ for all $i \in \mathcal{D}$.

Consider a lottery solution that is unbiased and stays within department quota. Let z_{ij}^t denote the the outcome of such lottery for category j in department i. Note that, by definition of such lottery, $Pr(z_{ij}^t = 1) = \alpha$ and $Pr(z_{ij}^t = 0) = 1 - \alpha$ must fold for all $i \in \mathcal{D}$. And, by definition, the random variable $z_{m+1,j}^t = \sum_{i=1}^{m} z_{i,j}^t$ is a sum of independent Bernoulli trials. Hence, $z_{m+1,j}^t$ has a binomial distribution. That is,

$$Pr(z_{m+1,j}^t = c) = \binom{m}{c} \alpha^c (1-\alpha)^{m-c}$$

.

Let $B_\alpha(m,\lambda)$ be the (upper) tail of the binomial distribution from λm to m. That is,

$$B_\alpha(m,\lambda) = \sum_{c=\lambda m}^{m} \binom{m}{c} \alpha^c (1-\alpha)^{m-c}$$

where λm is an integer and $\alpha < \lambda < 1$. When $\lambda = (1+\epsilon)\alpha$, by definition, the probability of $z^t_{m+1,j}$ is at least $b + x^t_{m+1,j} = (1+\epsilon)\mu$ is

$$B := Pr(z^t_{m+1,j} \geq (1+\epsilon)\mu) = B_\alpha(m,(1+\epsilon)\alpha).$$

The goal is to show that B is at least $e^{-\mu\epsilon^2 l}$, where $l > 0$ is a constant independent of μ and ϵ. This would imply that $f(\mu,\epsilon\mu) \geq e^{-\mu\epsilon^2 l}$. Hence, setting k to be any constant larger than l would prove the proposition.

To show lower bounds on the tail distribution, we use the following lemma.

Lemma 3. *Ahle (2017). When $\lambda \geq 0.5$,*

$$B_\alpha(m,\lambda) \geq \frac{1}{\sqrt{2m}} e^{-mH(\lambda;\alpha)}$$

where $H(\lambda;\alpha) = \lambda \log \frac{\lambda}{\alpha} + (1-\lambda)\log\frac{1-\lambda}{1-\alpha}$.

Applying this lemma for $m = \mu/\alpha$ and $\lambda = (1+\epsilon)\alpha$ implies:

$$\begin{aligned}
B &\geq \frac{1}{\sqrt{2\mu/\alpha}} e^{-\frac{\mu}{\alpha}H((1+\epsilon)\alpha;\alpha)} \\
&= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\frac{\mu}{\alpha}[(1+\epsilon)\alpha\log(1+\epsilon)+(1-(1+\epsilon)\alpha)\log\frac{1-(1+\epsilon)\alpha}{1-\alpha}]} \\
&= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu[(1+\epsilon)\log(1+\epsilon)+\frac{1-(1+\epsilon)\alpha}{\alpha}\log(1-\alpha\epsilon/(1-\alpha))]} \\
&\geq \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu[(1+\epsilon)\epsilon+\frac{1-(1+\epsilon)\alpha}{1-\alpha}\epsilon)]} && (1) \\
&= \frac{1}{\sqrt{2\mu/\alpha}} e^{-\mu\epsilon^2(1+\frac{1}{\epsilon}+\frac{1-(1+\epsilon)\alpha}{(1-\alpha)\epsilon})} && (2)
\end{aligned}$$

where (1) holds since $\log(1+\epsilon) < \epsilon$ and $\log(1-\alpha\epsilon/(1-\alpha)) < -\frac{\alpha\epsilon}{1-\alpha}$ for all $\epsilon \in (0,1)$.

The proof is complete when we observe that the right-hand side of (2) is larger than $e^{-\mu\epsilon^2 l}$ for any $l \geq 1 + 2/\underline{\epsilon}$ and sufficiently large μ.[20]

■

Proof of Proposition 1

In this section, we present the complete proof of Proposition 1. The proof is an adaptation of the procedure of Cox (1987).[21]

Proof. We present a constructive proof of Proposition 1 using following algorithm. The rounding algorithm takes a fair share table as input and generates a (random) reservation table as output. To make the algorithm easier to understand, after each step we demonstrate the algorithm on an example depicted in Figure 10.

Rounding Algorithm

Step 1: Given a fair share table X, we construct an extended table V by adding an extra row to table X. The last row of V is generated by taking 1 - fraction part of the column totals of table X.

In our example, shown in Figure 10, table V is equivalent to table X except the last row. Adding this extra row makes the column totals integers.

Figure 10: STEP 1 OF PROCEDURE

$$X = \begin{array}{ccc|c} 0.5 & 0.5 & 1 & 2 \\ 0.25 & 0.25 & 0.5 & 1 \\ 0.75 & 0.75 & 1.5 & 3 \\ \hline 1.5 & 1.5 & 3 & 6 \end{array}$$

(a) FAIR SHARE TABLE

$$V = \begin{array}{ccc|c} 0.5 & 0.5 & 1 & 2 \\ 0.25 & 0.25 & 0.5 & 1 \\ 0.75 & 0.75 & 1.5 & 3 \\ 0.5 & 0.5 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array}$$

(b) EXTENDED TABLE

[20] The proof is symmetric for the lower tail since $Pr(z^t_{m+1,j} \leq (1-\epsilon)\mu) = B_{1-\alpha}(m,(1-\epsilon)\alpha)$

[21] An alternative proof utilizes network flow approach, very similar to the one in proof Theorem 1. However, for its simplicity and ease of use by hand, we show a modified version of the procedure of Cox (1987).

We focus on the internal entries of table V. The procedure involves iterative adjustment of the fractions in table V until all fractions have been replaced by integers.

Step 2: If table V contains no fractions, then skip to Step 8.

Step 3: Choose any fraction v_{ij} in table V. At (i, j) begin an alternating row-column (or column-row) path of fractions. A cycle will be formed (all edges

Figure 11: STEP 3 OF PROCEDURE

$$V = \begin{array}{ccc|c} 0.5 & 0.5 & 1 & 2 \\ 0.25 & 0.25 & 0.5 & 1 \\ 0.75 & 0.75 & 1.5 & 3 \\ 0.5 & 0.5 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array}$$

fractions).

In our example, shown in Figure 11, the cycle of fractions is $(i_1, j_1) \to (i_1, j_2) \to (i_2, j_2) \to (i_2, j_3) \to (i_3, j_3) \to (i_3, j_2) \to (i_4, j_2) \to (i_4, j_1) \to (i_1, j_1)$.

Step 4: Modify the cycle. First, raise the odd edges and reduce the even edges at the same rate until at least one edge reaches an integer value. The resulting table then gives rise to a table V_1.

In our example, the odd edges rise by 0.5 and even edges reduce by 0.5 ($d_+ = 0.5$). The resulting table V_1 is shown in Figure 12.

Step 5: Next, readjust the edges in the cycle in the reverse direction, raising even edges and reducing odd edges in an analogous manner, which gives rises to another table V_2.

In our example, the even edges rise by 0.25 and odd edges reduce by 0.25 ($d_- = 0.25$). The resulting table V_2 is shown in Figure 12.

Step 6: Select either V_1 or V_2 with probabilities $p_1 = \frac{d_-}{d_- + d_+}$ and $\frac{d_+}{d_- + d_+}$, respectively.

In our example, table V is decomposed into table V_1 and table V_2 where

Figure 12: STEP 6 OF PROCEDURE

$$V_1 = \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0.25 & 0.75 & 0 & 1 \\ 0.75 & 0.25 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array} \qquad V_2 = \begin{array}{ccc|c} 0.25 & 0.75 & 1 & 2 \\ 0.25 & 0 & 0.75 & 1 \\ 0.75 & 1 & 1.25 & 3 \\ 0.75 & 0.25 & 0 & 1 \\ \hline 2 & 2 & 3 & 7 \end{array}$$

$V = \frac{1}{3}V_1 + \frac{2}{3}V_2$. There are few fraction elements in both tables.

Step 7: Reiterate Step 6 until no fractional elements left.

Step 8: Delete the last row of the table and report it as the outcome of the algorithm.

The algorithm must end in finite steps (at most the number of fractions in share table V) and, at the end we must have an integer table.

Lemma 4. *The outcome of the Rounding Algorithm stays within quota.*

Proof. In Step 4 and 5, after each adjustment the row and column sums remains the same. Moreover, after adjustments every element v_{ij} in table V always remains less than or equal to $\lceil v_{ij} \rceil$ and greater than or equal to $\lfloor v_{ij} \rfloor$. Therefore, the outcome of the algorithm will stay within quota. ■

Lemma 5. *The Rounding Algorithm satisfies the following property: For any iteration and for any entry of the table,*

$$E(v_{ij}|V) = v_{ij}$$

Proof. Note that in Step 4, v_{ij} raises by d_+ and in Step 5, it reduces by d_-. In Step 6, the probabilities of raising and decreasing are assigned as $\frac{d_-}{d_-+d_+}$ and $\frac{d_+}{d_-+d_+}$. Therefore, the expected adjustment will be $d_+\frac{d_-}{d_-+d_+} + d_-\frac{d_+}{d_-+d_+} = 0$. ■

In words, Lemma 5 proves that entries of the fair share table X are rounded up or down so that ex-ante positive and negative biases balance to yield zero bias.

Lemma 4 and Lemma 5 prove Proposition 1. ■

Proof of Proposition 2 and Proposition 3

Since Proposition 2 is a special case of Proposition 3, we prove the latter. We prove the proposition by contradiction.

Proof. Suppose a deterministic solution R stays within university quota. We show an example of a problem of reservation in two dimensions that the solution R can not have a finite bias. That is, for any constant $b > 0$, there exist a Y^t and an internal entry y^t such that $|\text{bias}(R(y^t))| > b$.

Example 3. *Consider a problem with three departments d_1, d_2, and d_3, two categories c_1, c_2, the reservation scheme vector $\boldsymbol{\alpha} = [0.5, 0.5]$. The departments d_1, d_2, and d_3 have $\mathbf{q}^1 = [0, 0, 1]$ positions in period-1 and $\mathbf{q}^2 = [1, 0, 0]$ positions in period-2.*

Notice that staying within university quota is equivalent to reserving exactly k positions for c_1 and c_2 in every 2k cumulative sum of vacancies in the university, where $k = 1, 2, 3, \ldots$. In period-1, department d_3 can reserve positions to either categories. Without loss of generality, we assume that it reserves 1 position for c_1. In period-2, since there are 2 cumulative sum of vacancies in the university, there should be exactly 1 position reserved for c_1. Department d_1 should reserve 1 position for category c_2. The period-1 and period-2 reservation tables are shown in Figure 13.

Figure 13: PERIOD-1 AND PERIOD-2 RESERVATION TABLES

$$X^1 = \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 1 \\ \hline 0.5 & 0.5 & 1 \end{array}$$

(a) PERIOD-1 FAIR SHARE TABLE

$$X^2 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

(b) PERIOD-2 FAIR SHARE TABLE

$$R(Y^1) = \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 1 \end{array}$$

(c) PERIOD-1 RESERVATION TABLE

$$R(Y^2) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

(d) PERIOD-2 RESERVATION TABLE

If departments have $\mathbf{q}^3 = [0,0,1]$ *positions in period-3, department* d_3 *can reserve its position to either categories. These two cases are show in Figure 15.*

Figure 14: TWO CASES FOR PERIOD-3 RESERVATION TABLES

$$X^3 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 1.5 & 1.5 & 3 \end{array}$$

(a) PERIOD-3 FAIR SHARE TABLE

$$R_1(Y^3) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \\ \hline 2 & 1 & 3 \end{array}$$

(b) PERIOD-3 RESERVATION TABLE

$$R_2(Y^3) = \begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 1 & 2 & 3 \end{array}$$

(c) PERIOD-3 RESERVATION TABLE

Case 1: We assume that the solution is $R = R_1$*. If the departments have* $\mathbf{q}^4 = [1,0,0]$ *positions in period-4, department* d_1 *should reserve 1 position for category* c_2*. Otherwise, the solution* R *would violate staying within university quota property. Period-4 fair share table and the period-4 reservation table are illustrated by* X_1^4 *and* $R_1(X_1^4)$ *in Figure 15.*

Case 2: We assume that the solution is $R = R_2$*. If the departments have* $\mathbf{q}^4 = [0,1,0]$ *positions in period-4, department* d_2 *should reserve 1 position for category* c_1*. Otherwise, the solution* R *would violate staying within university quota property. Period-4 fair share table and the period-4 reservation table are illustrated by* X_2^4 *and* $R_2(X_2^4)$ *in Figure 15.*

Figure 15: TWO CASES FOR PERIOD-4 RESERVATION TABLES

$$X_1^4 = \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(a) CASE 1: PERIOD-4 FAIR SHARE TABLE

$$R_1(X_1^4) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(b) CASE 1: PERIOD-4 RESERVATION TABLE

$$X_2^4 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(c) CASE 2: PERIOD-5 FAIR SHARE TABLE

$$R_2(X_2^4) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ \hline 2 & 2 & 4 \end{array}$$

(d) CASE 2: PERIOD-4 RESERVATION TABLE

If departments have $\mathbf{q}^5 = [0,0,1]$ *positions in period-3, department* d_3 *can reserve its position to either categories. These two cases are show in Figure 16.*

In Example 3 for each case, period-5 reservation for category c_1 in department d_1 is 0 and

Figure 16: TWO CASES FOR PERIOD-5 RESERVATION TABLES

$$X_1^5 = \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1.5 & 1.5 & 3 \\ \hline 2.5 & 2.5 & 5 \end{array}$$

(a) PERIOD-5 FAIR SHARE TABLE

$$R_{1.1}(Y_1^5) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \\ \hline 3 & 2 & 5 \end{array}$$

(b) PERIOD-5 RESERVATION TABLE

$$R_{1.2}(Y_1^5) = \begin{array}{cc|c} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 3 \\ \hline 2 & 3 & 5 \end{array}$$

(c) PERIOD-5 RESERVATION TABLE

$$X_2^5 = \begin{array}{cc|c} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 1.5 & 1.5 & 3 \\ \hline 2.5 & 2.5 & 5 \end{array}$$

(d) PERIOD-5 FAIR SHARE TABLE

$$R_{2.1}(Y_2^5) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \\ \hline 3 & 2 & 5 \end{array}$$

(e) PERIOD-5 RESERVATION TABLE

$$R_{2.2}(Y_2^5) = \begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \\ \hline 2 & 3 & 5 \end{array}$$

(f) PERIOD-5 RESERVATION TABLE

period-5 reservation for category c_2 in department d_2 is 0. We can extend these example for more periods analogously. The idea is following. In each period, the university has only one position. Department d_3 has always one position in odd periods and in the following period either department d_1 or department d_2 has one position according to these following cases.

- Case I: If department d_3 reserves 1 position to category c_1, department d_1 has one position in the next period.
- Case II: If department d_3 reserves 1 position to category c_2, department d_2 has one position in the next period.

In case I, department d_1 should reserve 1 position for category c_2, otherwise, solution would violate staying university quota property. In case II, department d_2 should reserve 1 position for category c_1, otherwise, solution would violate staying university quota property. Example 3 shows that if a solution stays within university quota, departments can grow in size without giving a seat to one category, i.e., the solution violates finite bias.[22]

This proves the proposition. ■

[22] An example for any number of categories and departments can be constructed in a similar way. Example 3 is constructed so that it not only illustrates the failure, but it also demonstrates any solution can fail to have finite bias in all categories.

B Tables and Figures

Figure 17: 200-point Roster prescribed by Government of India

FOR DIRECT RECRUITMENT

Model Roster of Reservation with reference to posts for Direct recruitment on All India Basis by Open Competition

Sl. No. of Post	Share of Entitlement				Category for which the posts should be earmarked
	SC @15%	ST @7.5%	OBC @27%	EWS @10%	
1	0.15	0.08	0.27	0.10	UR
2	0.30	0.15	0.54	0.20	UR
3	0.45	0.23	0.81	0.30	UR
4	0.60	0.30	1.08	0.40	OBC-1
5	0.75	0.38	1.35	0.50	UR
6	0.90	0.45	1.62	0.60	UR
7	1.05	0.53	1.89	0.70	SC-1
8	1.20	0.60	2.16	0.80	OBC-2
9	1.35	0.68	2.43	0.90	UR
10	1.50	0.75	2.70	1.00	**EWS-1**
11	1.65	0.83	2.97	1.10	UR
12	1.80	0.90	3.24	1.20	OBC-3
13	1.95	0.98	3.51	1.30	UR
14	2.10	1.05	3.78	1.40	ST-1
15	2.25	1.13	4.05	1.50	SC-2
16	2.40	1.20	4.32	1.60	OBC-4
17	2.55	1.28	4.59	1.70	UR
18	2.70	1.35	4.86	1.80	UR
19	2.85	1.43	5.13	1.90	OBC-5
20	3.00	1.50	5.40	2.00	SC-3
21	3.15	1.58	5.67	2.10	**EWS-2**
22	3.30	1.65	5.94	2.20	UR
23	3.45	1.73	6.21	2.30	OBC-6
24	3.60	1.80	6.48	2.40	UR
25	3.75	1.88	6.75	2.50	UR
26	3.90	1.95	7.02	2.60	OBC-7
27	4.05	2.03	7.29	2.70	SC-4
28	4.20	2.10	7.56	2.80	ST-2
29	4.35	2.18	7.83	2.90	UR
30	4.50	2.25	8.10	3.00	OBC-8
31	4.65	2.33	8.37	3.10	**EWS-3**

Source: https://dopt.gov.in/sites/default/files/ewsf28fT.PDF

Figure 18: 13-point Roster prescribed by Government of India

FOR DIRECT RECRUITMENT

Roster for Direct Recruitment otherwise than through Open Competition for cadre strength upto 13 posts

Cadre Strength	Initial Recruit-ment	Replacement No.												
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th
1	UR	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST
2	UR	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST	
3	UR	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST		
4	OBC	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST			
5	UR	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST				
6	UR	SC	OBC	UR	**EWS**	UR	OBC	SC	ST					
7	SC	OBC	UR	**EWS**	UR	OBC	SC	ST						
8	OBC	UR	**EWS**	UR	OBC	SC	ST							
9	UR	**EWS**	UR	OBC	SC	ST								
10	**EWS**	UR	OBC	SC	ST									
11	UR	OBC	SC	ST										
12	OBC	SC	ST											
13	SC	ST												

Source: https://dopt.gov.in/sites/default/files/ewsf28fT.PDF