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Keywords: Finance, Robust optimization, One-way trading, Fixed cost, Mini-max regret
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Upper and Lower Bounds on Robust One-way Trading with Fixed Costs

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Abstract

This paper considers the one-way trading problem with fixed costs where the trader can only trade in one direction throughout, either sell or buy, and he only knows limited information on price fluctuations beforehand. We construct a robust optimization model based on Savage's regret criterion, in order to find the online trading policy that minimizes the worst-case regret. However, it is very difficult to obtain analytical results if the trading horizon is relatively long, due to the discontinuity in the trader's objective function caused by the fixed cost. Thus we propose to solve the alternative problem with prepaid trading opportunities, which is not only a satisfactory approximation of the original one, but also a realistic problem with many practical applications, such as in the stock or future market. The optimal online trading policy of the new problem can be easily found based on the existing results of the one-way trading problem with limited opportunities in Wang and Lan (2019). The proposed trading policy is robust in that it guarantees a finite performance gap between itself and the optimal offline trading policy, no matter how prices fluctuate within the given range. It is proved that this gap is an upper bound on the minimal CD of the original problem with fixed costs. A lower bound on the minimal CD of the original

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problem is also provided.

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1. Introduction

One-way trading refers to trading a given quantity of goods in one direction (either selling or buying) within a finite time horizon when the price fluctuates continuously. It is a fundamental activity in daily economy, such as stock trading, currency exchange, merchandising, inventory procurement and so on. The basic one-way trading problem without transaction costs has been well studied in the literature (El-Yaniv et al. (2001), Chen et al. (2001), Zhang et al. (2012), Wang et al. (2016)). Such a simple set-up of the one-way trading problem is very elegant from a theoretical point of view; however, in real word applications, transaction costs are ubiquitous. Take the stock trading process as an example, there are two types of transaction costs, variable costs and fixed costs. Since the variable costs are proportional to the transaction volume, such as stamp duty, commission and so on, they can be easily handled by proportionally adjusting the price. The fixed costs, on the other hand, are paid for each transaction, regardless of the trading volume. The existence of fixed costs fundamentally changes the structure of the basic one-way trading problem and asks for further research.

Since the price fluctuates during the entire trading horizon, the information about future prices plays a crucial role in the one-way trading problem. If all the future prices were known in advance, the trader simply trades everything at the best price in the entire trading horizon, which is known as the optimal offline trading policy. Unfortunately, information about future prices is never perfect in reality. One-way trading is usually regarded as an online decision-making process, because the irrevocable transaction under the current price is committed without knowing the price in the future. There are two possible mistakes due to imperfect information: trading too little when current price is quite favorable or trading too much when it is not so favorable. Some researches in the one-way trading literature assume that the trader knows the probability distribution on price that can be estimated from historical data. However, due to the lack of historical data, or the non-stationary fluctuation of prices, it's often unreliable if not impossible to estimate the probabilistic distribution of future prices. Therefore, it is desirable for us to study the one-way trading problem with limited yet potentially more reliable information on future prices.

This paper solves the one-way trading problem with fixed costs where the trader only knows the range of future prices. The same competitive difference (CD) analysis is employed as by Wang et al. (2016) for the basic one-way trading problem without fixed costs. CD analysis aims at finding the online trading policy that minimizes the worst-case regret. In view of the calculation complexity caused by the fixed cost in the original problem, we propose to solve the alternative problem with prepaid limited opportunities instead. The one-way trading problem with limited opportunities is not only a satisfactory approximation of the original problem with fixed costs, but also a realistic problem with many practical applications. Putting a limit on trading opportunities is beneficial from the perspectives of both practitioners and scholars in the trading of financial products, such as stocks or futures. A famous piece of advice to all investors from Warren Buffett is to be patient and avoid frequent trading. Interestingly, many empirical researches in the academia seem to support this investment wisdom (Odean (1999), Barber and Odean (2000), Cohn-Urbach and Westerholm (2006)). Based on the existing results of the one-way trading problem with limited opportunities in Wang and Lan (2019) , it is not difficult to obtain the online trading policy and the corresponding CD of the problem with prepaid opportunities (denoted by D^{PO}). It is proved that D^{PO} is an upper bound on the minimal CD of the original problem with fixed costs. In addition, a lower bound on the minimal CD of the original problem is also provided in this paper, which serves as a benchmark to evaluate the proximity of the performances between the problem with prepaid opportunities and the original one with fixed costs.

Another contribution of this paper is that we propose a reliable method to approximately solve these complicated robust optimization problem involving multiple decision epochs and multiple state variables. It is often difficult to solve this type of multi-period zero-sum game between the decision maker and the adversary through backward induction directly. Because the subcases which require comparison in each decision epoch increases dramatically when the backward process goes on. In this case, we can purposely reduce the strategy space of the decision maker and solve the transformed problem instead, which provides a heuristic solution to the original problem and an upper bound on the objective of the original problem. On the other hand, by purposely reducing the strategy space of the adversary, we can figure out the lower bound on the objective of the original problem and use it to evaluate the performance of the heuristic solution.

The rest of the paper is organized as follows. In Section 2, we review the related literature, especially those published in recent years. In Section 3, we start the analysis by formulating the theoretical models. In Section 4, we provide theoretical analysis of the problem and derive closed-form solution for the two period problem. In Section 5, we develop the upper and lower bounds and conduct numerical experiments. We provide managerial insights and possible extensions for future study in the concluding section.

2. Literature Review

Earlier studies on the one-way trading problems take the Bayesian approach with probabilistic information on the price, and provide optimal trading policies based on Bayes' rule. In this regard there is already an extensive review provided by Lippman and McCall (1981), to which we refer any interest in that stream of research. We focus on the more recent researches that employ robust optimization with limited information, which are more close to our current study. We will also look into the literature on the time series search problem, which is a special case of the problem we study in this paper. A brief discussion on the methodology is also furnished.

The traditional one-way trading problem, that is, the one without limit on trading opportunities, is a special case with limited trading opportunities when the limit is large enough. El-Yaniv et al. (2001) are the first to consider the one-way trading problem with limited information where the trader only knows the bounds of price. They find the threat-based online trading policy via competitive ratio (CR) analysis proposed by Sleator and Tarjan (1985). Chen et al. (2001) and Zhang et al. (2012) investigate the one-way trading problem in an interesting setting where the range of the price in each period depends geometrically on the price in the previous period. They design the optimal static online trading strategy based on CR analysis. Larsen and Wøhlk (2010) study the optimal online inventory control policy via CR analysis with consideration of inventory holding cost and fixed order cost. Dai et al. (2016) improve the upper bound of the CR given by Larsen and Wøhlk (2010) in the case with order costs and bounded storage capacity. To overcome the calculation difficulties caused by the parameter of fixed order cost, they bound the CR for the two types of cost separately before making a balance between them. Also, they utilize the intuition as previous CR analyses did. In comparison, to avoid the complexity of analyzing the fixed cost directly, we deal with a closely related problem —- the one-way trading problem with limited opportunities. Also, the method of CD analysis we adopted enables us to get the analytical results without the need of any intuition beforehand.

The time series search problem is also a special case of the one-way trading problem with limited trading opportunities, when the limit is set to one. Different variants of the search problem have been studied via CR analysis. El-Yaniv et al. (2001) provide the optimal deterministic and randomized reservation price policies for the search problem with price bounds. Damaschke et al. (2009) propose an optimal deterministic trading policy and a near-optimal randomized trading policy for the search problem with upper and lower price bounds decreasing in time. They also find an optimal randomized trading policy for the search problem with only the upper price bound decreasing in time. Xu et al. (2011) generalize the search problem by introducing a profit function that is increasing in price and decreasing in time. They give the optimal deterministic trading policy respectively when duration is either known or unknown in advance. A direct extension of the basic search problem is the k-search problem, where k indivisible products must be traded within a time horizon. Lorenz et al. (2009) propose an optimal strategy for the k-search problem where one can only trade one product at a time, while Zhang et al. (2011) allow the trading of multiple products at a time.

Although frequently employed as a robust optimization method to solve online problems with limited information, the CR analysis is often criticized for being too conservative. Some alternative quality measures of the online strategies to mitigate this have been proposed in the literature, which are reviewed by Dorrigiv and López-Ortiz (2005). A variety of performance indicators of the online search strategies are compared by Boyar et al. (2012). An average-case analysis is provided by Fujiwara et al. (2011) for the one-way trading in currency exchange where the upper bound of the exchange rate follows a random distribution. They derive and compare different optimal strategies for several different measures of performance. However, all these studies still adhere to competitive ratio as the objective, which often results in analysis difficulties. In order to do this, they also need more information, such as the distribution of the price bounds. Wang et al. (2016) approach the one-way trading problem with the CD analysis for the first time. Instead of the ratio, they use the difference in revenue or cost between the online trading policy and the optimal offline trading policy as the objective. Generally speaking, the CD analysis is not only more amenable to analysis, but also less conservative, remedying the shortcomings of the CR analysis to some extent.

3. Problem Description and The Model

In this section we give details on how we formulate the problem and apply the CD analysis. As there is no structural difference between the selling and buying problem, from now on our analysis will focus on the former, while the results can be easily transferred to the latter.

3.1. Problem Description

We consider the one-way trading problem within a finite time horizon consisting of $T(T\geq 2)$ discrete periods. Let Q designate the total volume of the goods to be sold out during the horizon. Assume the goods for trading is fully divisible, such as gasoline, coal, steel, etc., which can be partially traded at will. Without loss of generality, the total volume of the goods is normalized to one, i.e., $Q = 1$, and the measure of the price is adjusted accordingly. It is known beforehand that the price fluctuates within a particular range $[m, M]$, and nothing about the probability density on this range is known. In the beginning of each period t, a price $p_t \in [m, M]$ is revealed and it holds constant during that entire period. On the observation of p_t, the trader must decide on the trading volume k_t in period t before this price expires. In the last period, whatever the price, all the remaining goods must be sold out. There is no sampling cost to obtain the price quotation. All variable costs can be contained in the price and thus are not explicitly considered in the model for simplicity. A fixed fee, denoted by c, must be paid for each transaction.

This one-way trading problem is a typical online decision-making problem for the trader, since he must make trading decisions based on currently known information. The main objective of this paper is to find an effective online trading policy for the trader. We only focus on deterministic trading polices, as they are usually employed in practice, while randomized trading policies are more of theoretical interests. A deterministic online trading policy determines for each $t = 1, 2, \ldots, T$ a unique trading volume k_t that depends on the current price p_t and the historical transaction information $H_t = (p_1, p_2, \ldots, p_{t-1}; k_1, k_2, \ldots, k_{t-1})$. A formal way to describe such an online policy $\mathcal{A}$ is by a series of functions£¬

$$\mathcal{A} = \{k_t(p_t, H_t), t = 1, ..., T\} \tag{1}$$

where $\sum_{t=1}^{T} k_t(p_t, H_t) = 1$ must hold to ensure the completion of the trading task during the finite horizon. Next we discuss the criterion to choose an online trading policy, assuming the trader certainly prefers more sales revenue.

3.2. The Model

We adopt Savage's minimax regret as a performance measure of the online trading policy $\mathcal{A}$. The minimax regret criterion is first applied by Wang et al. (2016) to the one-way trading problem without any consideration about transaction costs. They called this approach the CD analysis, in contrast to the CR analysis well studied in the literature.

The regret of an online trading policy is the performance difference between itself and the optimal offline trading policy with perfect price information. For different price realizations, the regret would vary, and the worst regret is defined to be the CD of the online policy. Let $D(\mathcal{A})$ denote the CD of the online trading policy $\mathcal{A}$ in the sales problem, then

$$D(\mathcal{A}) \quad = \max_{\mathbf{p}} \quad [O^*(\mathbf{p}) - O(\mathbf{p}; \mathcal{A})], \tag{2}$$

where $O^*(\mathbf{p})$ is the revenue of the optimal offline trading policy while $O(\mathbf{p}; \mathcal{A})$ is that of the online trading policy $\mathcal{A}$ under a realized price path $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$. The objective of the CD analysis is to find an online policy $\mathcal{A}$ that minimizes $D(\mathcal{A})$. The basic idea of CD analysis is to find an online trading policy as close as possible to the optimal offline trading policy, where the degree of closeness is defined by the competitive difference (the worst regret).

The CD analysis can be regarded as a zero-sum game between the trader and the nature. The trader determines the online trading policy $\mathcal{A}$ that minimizes the regret $[O^*(\mathbf{p}) - O(\mathbf{p}; \mathcal{A})]$, while the nature chooses the price $\mathbf{p}$ to maximize the regret. Since the nature always picks the worst price for the trader, thus we call it the adversary henceforth. The timeline of the game is as follows. In any period t ($t = 1, 2, \ldots, T$), the adversary chooses a price p_t within the interval $[m, M]$ first to *maximize* the competitive difference; the trader then determines the trading volume k_t at this price to *minimize* the competitive difference. The value of the subgame between the trader and the adversary from period t through T, or the competitive difference from period t on, which is designated by D_t, only depends on the historical transaction information H_t. This information can be conveyed by the trading state s_{t-1} at the start of period t, which consists of three state variables, that is,

$$s_{t-1} \quad = \quad \{C_{t-1}, K_{t-1}, \bar{p}_{t-1}\} \tag{3}$$

where $C_{t-1} = \sum_{i=1}^{t-1} \left(p_i k_i - c \times 1_{\{k_i>0\}}\right)$ is the accumulated profit, $K_{t-1} = \sum_{i=1}^{t-1} k_i$ is the accumulated trading volume, and $\bar{p}_{t-1} = max(p_{1:t-1})$ is the highest price from period 1 through $t-1$. The state transition to s_t by actions taken in period t can be represented as

$$s_t \quad = \quad s_{t-1} + \{p_t k_t - c \times 1_{\{k_t>0\}}, k_t, \bar{p}_t - \bar{p}_{t-1}\} \tag{4}$$

In the beginning of any period $t \in \{1, 2, \ldots, T-1\}$, given the state s_{t-1}, we have

$$D_t(s_{t-1}) \quad = \quad \max_{p_t \in [m,M]} \; \min_{k_t \in [0, 1-K_{t-1}]} \; D_{t+1}(s_t) \tag{5}$$

where

$$D_T(s_{T-1}) = \max_{p_T \in [m,M]} \left\{ \underbrace{\max(\bar{p}_{T-1}, p_T) - c}_{\text{total offline profit}} - \underbrace{\left(C_{T-1} + p_T(1-K_{T-1}) - c \times 1_{\{1-K_{T-1}>0\}}\right)}_{\text{total online profit}} \right\} \tag{6}$$

Note that $k_T = 1 - K_{T-1}$ holds to guarantee that all the goods are sold out in the last period. Denote the one-way trading problem with fixed transaction cost as problem FC. The minimal CD of the trader can achieve in problem FC is the value of the game, that is,

$$D^{FC} = D_1(s_0) \tag{7}$$

where $s_0 = \{0, 0, m\}$ is the initial state at the beginning of the first period. Table 1 lists the notations used for key parameters and variables in the model.

Table 1: Notation of key parameters and variables

Symbol	Description
T	Length of the trading horizon, $T > 2$.
t	Period index, $t = 1, 2, \ldots, T$.
Q	The total volume of the goods to be sold out, and $Q = 1$.
m, M	Lower bound and upper bound on the prices in all periods.
p_t	Price revealed at period t.
k_t	Trading amount of the trader at period t, and $\sum_{t=1}^{T} k_t = 1$ must be ensured to complete the trading task.
H_t	The historical transaction information at the beginning of period t, $H_t = [(p_1, k_1), \ldots, (p_{t-1}, k_{t-1})]$.
K_t	Total volume traded by the end of period t, $K_t = \sum_{i=1}^{t} k_i$.
C_t	Total profit obtained by the end of period t, $C_t = \sum_{i=1}^{t} \left(p_i k_i - c \times 1_{\{k_i > 0\}}\right)$.
$\bar{p}_t$	The highest price seen by the end of period t, $\bar{p}_t = \max(p_1, p_2, \ldots, p_t)$.
s_t	The set of state variables at the beginning of period $t+1$, $s_t = \{C_t, K_t, \bar{p}_t\}$.
s_0	The initial state at the beginning of the first period, $s_0 = \{0, 0, m\}$.
p_t^*	The optimal price for the adversary at period t.
k_t^*	The optimal trading volume for the trader at period t.
$D_t(s_{t-1})$	The CD at period t with s_{t-1} given.
$B_t(K_{t-1}, \bar{p}_{t-1})$	The adjusted CD at period t with K_{t-1} and $\bar{p}_{t-1}$ given, which does not take the sunk cost C_{t-1} into consideration.

4. Preliminary Analysis of Problem FC

There are two properties for the model of Problem FC in (5) and (6), as stated in the following lemmas. The proofs are in the appendix.

Lemma 1. *If the price range $[m, M]$ is normalized to $[0, 1]$ by normalizing p_t and c to $p_t' = \frac{p_t - m}{M - m}$ and $c' = \frac{c}{M - m}$ respectively, then the CD of the normalized model is $\frac{D^{FC}}{M - m}$.*

Lemma 2. *The strategies of both the trader and the adversary in an arbitrary period t in problem FC are independent of C_{t-1}.*

According to Lemma 1, we can normalize the price range to $[0, 1]$ without losing any generality of the model. In addition, based on Lemma 2, the original model in (5) and (6) can be simplified to a new model with only two state variables, K_{t-1} and $\bar{p}_{t-1}$. Let $B_t(K_{t-1}, \bar{p}_{t-1})$ be *the adjusted CD* in period t with K_{t-1} and $\bar{p}_{t-1}$ given, which denotes the gap between the additional profit of the online policy from period t to the last period and the total offline profit. So, the relationship between the CD and the adjusted CD is $D_t(s_{t-1}) = B_t(K_{t-1}, \bar{p}_{t-1}) - C_{t-1}$.

After introducing $B_t(K_{t-1}, \bar{p}_{t-1})$, we can rewrite the game as follows. For any period $t \in \{1, 2, \ldots, T-1\}$, given the state K_{t-1} and $\bar{p}_{t-1}$, *the adjusted CD* is

$$B_t(K_{t-1}, \bar{p}_{t-1}) \;=\; \max_{p_t \in [0,1]} \; \min_{k_t \in [0, 1-K_{t-1}]} \; B_{t+1}(K_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}) \tag{8}$$

The boundary condition is

$$\begin{aligned} B_T(K_{T-1}, \bar{p}_{T-1}) &= \max_{p_T \in [0,1]} \left\{ \underbrace{\max(\bar{p}_{T-1}, p_T) - c}_{\text{total offline profit}} - \underbrace{\left(p_T(1-K_{T-1}) - c \times 1_{\{1-K_{T-1}>0\}}\right)}_{\text{additional online profit}} \right\} \\ &= max\left(K_{T-1}, \bar{p}_{T-1}\right) - c + c \times 1_{\{1-K_{T-1}>0\}} \end{aligned} \tag{9}$$

The minimal CD the trader can achieve in problem FC is

$$D^{FC} = B_1(K_0, \bar{p}_0) \tag{10}$$

Lemma 3 points out the monotonicity of $B_t(K_{t-1}, \bar{p}_{t-1})$ in K_{t-1}. It reveals that when there is more volume left, the trader will have a better chance to improve his performance in the remaining periods.

Lemma 3. *For any $t \in \{1, 2, \ldots, T\}$, $B_t(K_{t-1}, \bar{p}_{t-1})$ is weakly increasing in K_{t-1} when $K_{t-1} < 1$.*

Based on Lemma 3, we can derive that it is optimal for the trader to trade nothing when the adversary sets the current price at the bottom level m, unless it is the last period when the trader has to finish the trading task, as stated in Lemma 4.

Lemma 4. *In any period $t \in \{1, 2, \ldots, T-1\}$, if $p_t = m$, then it is optimal for the trader to trade noting, i.e., $k_t^* = 0$.*

There is also a particular feature with respect to the adversary's strategy when she chooses to lower the current price, as stated in Lemma 5.

Lemma 5. *In an arbitrary period t, if the adversary chooses to reduce the price in this period, then it is optimal for her to reduce it to the bottom price m and keep it at m in the subsequent periods.*

According to Lemma 5, we can reduce the adversary's strategy space while maintaining the same CD, which largely simplifies the analysis process. The adversary can only choose from the following two options when she sets the current price in an arbitrary period $t \in \{1, 2, \ldots, T\}$:

(a)increase the price to a level that is not lower than the previous price, that is, $p_t \geq p_{t-1}$; or

(b)drop the price to the bottom level, i.e., $p_t = 0$.

In other words, the adversary will either keep an weakly increasing price path, or drop the price to the bottom level permanently. She is not interested in vibrating the price. This problem with reduced strategy space for the adversary is called problem FC' hereafter. Therefore, if the adversary chooses to reduce the price to the bottom level 0, the trader sells all the remaining goods in one go to save fixed costs and the game ends in advance. Also, in any period t, if the previous price $p_{t-1} > 0$, it indicates that the adversary has been keeping the weakly increasing price path from period 1 to the previous period, thus $\bar{p}_{t-1} = p_{t-1}$. Since the only difference between Problem FC' and Problem FC is the strategy space of the adversary, the model of problem FC in (8) and (9) can be easily revised to describe Problem FC' as follows. In any period $t \in \{1, 2, \ldots, T\}$ with the state

K_{t-1} and $\bar{p}_{t-1} = p_{t-1} > 0$, the adjusted CD is

$$\begin{aligned} &\breve{B}_t(K_{t-1}, p_{t-1}) \\ &\qquad = \max\left\{p_{t-1} - c - \left(0 - c \times 1_{\{1-K_{t-1}>0\}}\right), \max_{p_t \in [p_{t-1},1]} \breve{A}_t(K_{t-1}, p_{t-1}|p_t)\right\} \end{aligned} \tag{11}$$

where $p_{t-1} - c - \left(0 - c \times 1_{\{1-K_{t-1}>0\}}\right)$ is the CD when $p_t = 0$ while

$$\breve{A}_t(K_{t-1}, p_{t-1}|p_t) \quad = \min_{k_t \in [0,1-K_{t-1}]} \breve{B}_{t+1}(K_t, p_t) - (p_t k_t - c \times 1_{\{k_t>0\}}) \tag{12}$$

is the minimal CD for the trader when $p_t \geq p_{t-1}$. The boundary condition is

$$\begin{aligned} \breve{B}_T(K_{T-1}, p_{T-1}) &= \max_{p_T \in \{0\} \cup [p_{T-1},1]} \left\{ \underbrace{\max(p_{T-1}, p_T) - c}_{\text{total offline profit}} - \underbrace{\left(p_T(1 - K_{T-1}) - c \times 1_{\{1-K_{T-1}>0\}}\right)}_{\text{additional online profit}} \right\} \\ &= \max(K_{T-1}, p_{T-1}) - c + c \times 1_{\{1-K_{T-1}>0\}} \end{aligned} \tag{13}$$

The minimal CD the trader can achieve in problem FC' is

$$D^{FC'} = \breve{B}_1(K_0, p_0) \tag{14}$$

The relationship between problem FC' and problem FC is stated in Theorem 1. The monotonicity of $\breve{B}_t(K_{t-1}, p_{t-1})$ is illustrated in Lemma 6. The proofs are given in the appendix.

Theorem 1. *The CDs of problem FC and problem FC' are the same, i.e.,* $D^{FC} = D^{FC'}$.

Lemma 6. *For any* $t \in \{1, 2, \ldots, T\}$, $\breve{B}_t(K_{t-1}, p_{t-1})$ *is weakly increasing in* K_{t-1} *when* $K_{t-1} < 1$.

Theoretically, the sequential zero-sum game can be solved via backward induction. We can obtain the closed-form solutions for the two-period ($T = 2$) problem by this method, which is summarized in Theorem 2 , and the proof is given in the appendix.

Theorem 2. *For the two-period ($T = 2$) one-way trading problem with fixed costs c, the optimal online trading policy for the trader is as follows: In the first period, given p_1, the optimal trading volume is*

$$\hat{k}_1(p_1) = \begin{cases} 0 & if\ p_1 \leq p_{11} \\ p_1 & if\ p_{11} < p_1 < p_{12} \\ 1 & if\ p_1 \geq p_{12} \end{cases} \tag{15}$$

where $p_{11} = min(1/2, \sqrt{c})$ and $p_{12} = max(1/2, 1 - \sqrt{c})$; In the second period, the trading volume is $1 - \hat{k}_1(p_1)$, no matter what the price p_2 is. The CD of this online trading policy is $D^{FC} = \min(1/4 + c, 1/2)$.

When the trading horizon becomes longer ($T \geq 3$), the analytical complexity of the backward induction process increases dramatically, mainly resulting from the discontinuity of the objective function with respect to the trading volume in the inner minimization problem in (8). In any period $t = 1, 2, \ldots, T-1$, there are two discontinuous points on the objective function $B_{t+1}(K_t, \bar{p}_t)$ for the trader, which are $k_t = 0$ and $k_t = 1 - K_{t-1}$. Worse still, the continuous part of $B_{t+1}(K_t, \bar{p}_t)$ on the open interval $(0, 1 - K_{t-1})$ is also very complicated because of the discontinuities in subsequent periods. It is a piecewise function with several local minima, all of which are candidates for the optimal trading volume. The number of the segments consisting the continuous part of $B_{t+1}(K_t, \bar{p}_t)$ increases when the backward process goes on, so does the number of the local minima. As a result, if the trading horizon is relatively long (in other words, when T is large), it is very difficult if not impossible to obtain analytical results directly via backward induction. Therefore, we will transform problem FC into a more solvable approximate problem—- the one-way trading problem with prepaid opportunities (hereafter called Problem PO) in the next section. The objective value of Problem PO (denoted by D^{PO}) serves as a upper bound on D^{FC}, while its optimal online trading policy can be used as a heuristic of Problem FC. In order to evaluate the proximity between D^{PO} and D^{FC}, we also provide a lower bound on D^{FC}.

5. The Upper and Lower Bounds

In order to find both the upper and lower bound on D^{FC}, we need the following lemma and the corollary derived directly from it. The proof of Lemma 7 is in the appendix.

Lemma 7. *If $f_1(x,y) \le f_2(x,y)$, and $X' \subseteq X$, $Y' \subseteq Y$, then*

$$\max_{x \in X'} \min_{y \in Y} f_1(x,y) \le \max_{x \in X} \min_{y \in Y'} f_2(x,y) \tag{16}$$

Each side of the inequality (16) can be regarded as a zero-sum game between a maximizer and a minimizer where each player makes only one decision. The strategy space of the maximizer is constrained on the left-hand side of (16), while the strategy space of the minimizer is constrained on the right-hand side of it. Lemma 7 can be generalized to apply in the multiple-stage zero-sum game where both players take turns to make decisions for several times. The generalized result is formally stated in Corollary 1, which can be easily proved by applying Lemma 7 repeatedly.

Corollary 1. *Let $x = \{x_1, x_2, \ldots, x_j\}$, $y = \{y_1, y_2, \ldots, y_j\}$. If $f_1(x,y) \le f_2(x,y)$, and $X_i' \subseteq X_i$, $Y_i' \subseteq Y_i$ for $i = 1, 2, \ldots, j$, then*

$$\begin{aligned} &\max_{x_1 \in X_1'} \min_{y_1 \in Y_1} \max_{x_2 \in X_2'} \min_{y_2 \in Y_2} \ldots \max_{x_j \in X_j'} \min_{y_j \in Y_j} f_1(x,y) \le \\ &\qquad \max_{x_1 \in X_1} \min_{y_1 \in Y_1'} \max_{x_2 \in X_2} \min_{y_2 \in Y_2'} \ldots \max_{x_j \in X_j} \min_{y_j \in Y_j'} f_2(x,y) \end{aligned} \tag{17}$$

One application of Corollary 1 is to analyze the monotonicity of the objective function of problem FC. The monotonicity of D^{FC} with respect to c is straightforward according to Corollary 1.

Lemma 8. *D^{FC} is increasing in the fixed cost c.*

Another application of Lemma 7 and Corollary 1 is to find the upper (lower) bound on D^{FC} by solving the variant of problem FC where the minimizer/trader's (maximizer/adversary's) strategy space is purposely reduced.

5.1. Lower bound

We provide a lower bound on $D^{FC'}$ or equally on D^{FC} by solving a discrete version of problem FC', which is called problem FC" hereafter. We evenly divide the interval of possible values for k_t into E sub-intervals, so $k_t \in \{e \times \Delta_k \ for \ e \in \{0, 1, \ldots, E\}\}$ where $\Delta_k \equiv 1/E$; we also evenly divide the interval of possible values for p_t into F sub-intervals, so $p_t \in \{f \times \Delta_p \ for \ f \in \{0, 1, \ldots, F\}\}$ where $\Delta_p \equiv 1/F$. As a result, both state variables K_{t-1} and p_{t-1} are also discrete-valued. Now, the continuous model of problem FC' can be transformed into the following discrete model of problem FC". We introduce two matrix with $(E+1)$ rows and $F+1$ columns for each period, $\dot{B}_t$ and $\dot{A}_t$, which are corresponding to $\breve{B}_t$ and $\breve{A}_t$ in the continuous model. $\dot{B}_t(e, f)$ designates the CD in period t given the state $K_{t-1} = e \times \Delta_k$ and $p_{t-1} = f \times \Delta_p$, while $\dot{A}_t(e, f)$ designates the minimal worst-regret given the state $K_{t-1} = e \times \Delta_k$ and the current price $p_t = f \times \Delta_p$. In any period $t \in \{1, 2, \ldots, T-1\}$ with the state $K_{t-1} = i \times \Delta_k$ and $\bar{p}_{t-1} = p_{t-1} = j \times \Delta_p > 0$, the CD is

$$\dot{B}_t(i, j) = \max\Big\{j \times \Delta_p - c + c \times 1_{\{i<E\}},\ \max\Big\{\dot{A}_t(i, f) \ for \ f = j, j+1, \ldots, F\Big\}\Big\} \quad (18)$$

where

$$\dot{A}_t(i, j) = \min\Big\{\dot{B}_{t+1}(i, j),\ \mathcal{G}_t(i, j)\Big\} - j \times \Delta_p \Delta_k \quad (19)$$

with $\mathcal{G}_t(i, j) = \min\Big\{\dot{B}_{t+1}(i+e, j) - j\Delta_p \times e\Delta_k + c \ for \ e = 1, 2, \ldots, E-i\Big\}$. (18) and (19) in the discrete model are corresponding to (11) and (12) in the continuous model. Note that there is an extra term "$-j \times \Delta_p \Delta_k$" in (19) compared with (12). The boundary condition in (13) becomes

$$\dot{B}_T(i, j) = \max\{i\Delta_k, j\Delta_p\} - c + c \times 1_{\{1-i\Delta_k>0\}} \quad (20)$$

in the discrete model. The CD of problem FC" is

$$D^{FC''} = \dot{B}_T(1,1). \tag{21}$$

Theorem 3 points out that the CD of the discrete model above is still a lower bound on D^{FC}, and the proof is in the appendix.

Theorem 3. *The CD of the discrete version of problem FC' in equations (18),(19) and (20) is still a lower bound on D^{FC}, i.e., $D^{FC''} \leq D^{FC'} = D^{FC}$.*

The following relationships between each two adjacent columns in $\dot{B}_t$, or between each two adjacent rows in $\dot{A}_t$ can help us greatly reduce the calculation complexity.

$$\begin{aligned}\mathcal{G}_t(i,j) &= \min\left\{\dot{B}_{t+1}(i+e,j) - j\Delta_p \times e\Delta_k + c \ for \ e = 1,2,\ldots,E-i\right\} \\ &= \min\left\{\dot{B}_{t+1}(i+1,j) - j\Delta_p\Delta_k + c, \ \ \mathcal{G}_t(i+1,j) - j\Delta_p\Delta_k\right\}\end{aligned} \tag{22}$$

$$\begin{aligned}\mathcal{Z}_t(i,j) &= \max\left\{\dot{A}_t(i,f) \ for \ f = j, j+1, \ldots, F\right\} \\ &= \max\left\{\dot{A}(i,j), \ \ \mathcal{Z}_t(i,j+1)\right\}\end{aligned} \tag{23}$$

The discrete version of problem FC' can be calculated very efficiently, as stated in Theorem 4.

Theorem 4. *The equilibrium outcome of the discrete model for problem FC' can be found in $O(T*E*F)$ time.*

5.2. Upper bound

To find the upper bound on D^{FC}, we introduce two relevant problems.

(i) Problem FC-LO: or the one-way trading problem with fixed costs and limited opportunities. Compared with Problem FC, in Problem FC-LO a new constraint is imposed on the trader that he can conduct at most N transactions during the trading horizon.

(ii) Problem PO: or the one-way trading problem with prepaid opportunities, where the trader first "buys" a fixed number of trading opportunities (denoted by N) in advance before the trading horizon starts and pays the fixed cost $N * c$. During the trading horizon, the trader can conduct at most N transactions. No money will return to the trader even if he trades less than N times in the end.

The relationships between the CD of these two problems and the CD of problem FC are summarized in Theorem 5, and the proof is in the appendix.

Theorem 5. *(Bounds on D^{FC}) $D^{FC'} = D^{FC} \leq D^{FC-LO} \leq D^{PO}$.*

We use D^{PO} to replace D^{FC-LO} as the upper bound on D^{FC}, because problem PO is more solvable than problem FC-LO.

In problem PO, the trader "buys" a fixed number of trading opportunities (denoted by N)and pays the fixed cost $N * c$ in advance before the trading horizon starts. So the minimal attainable CD for the trader in problem PO is

$$D^{PO} = \min_{N \leq T} \ (N-1)c + D^{LO}(N) \tag{24}$$

where $D^{LO}(N)$ is the minimal attainable CD for the trader in the one-way trading problem with N opportunities exogenously given. The one-way trading problem with limited opportunities (and no fixed cost) will be briefly called *problem LO* hereafter. Wang and Lan (2019) have studied problem LO, and their results about $D^{LO}(N)$ is given in the following lemma.

Lemma 9. *(Wang and Lan (2019)) For the T-period one-way trading problem with N opportunities, the minimal CD of all online trading policies is*

$$D^{LO}(N) = \begin{cases} \left(\frac{N-1}{N}\right)^N \equiv f_1(N) & if \ N = T \\ \left(\frac{N}{N+1}\right)^N \equiv f_2(N) & if \ N \leq T-1 \end{cases} \tag{25}$$

According to Lemma 9, if $N \leq T-1$, $D^{LO}(N)$ is irrelevant of T, and it decreases in N at a decreasing speed. In other words, increasing trading opportunities for the trader in Problem LO displays a diminishing marginal effect on the minimal CD. In addition, an important observation for problem LO given by Wang and Lan (2019) is that, the number of actually used opportunities in equilibrium can be any integer in the interval $[1, N]$, which is determined by the trader's decisions. The more opportunities used, the higher the revenue achieved by both the offline and online trading policies. However, the revenue gap between the online and offline policies (i.e., the CD) is not influenced by the number of actually used opportunities. It is worth mentioning that, in Problem PO, the prepaid fixed costs is always $c \times N$, although the actual number of transactions might turn out to be less than N.

Based on Lemma 9, we can obtain the objective function in the right-hand-side of (24) (i.e., $\mathcal{H}(N) = (N-1)c + D^{LO}(N)$), which enables us to examine the optimal number of trading opportunities for the trader to prepaid for (designated by N^*) and thus figure out D^{PO}. The function $\mathcal{H}(N)$ may not be monotonic in N. When $1 \leq N \leq T-1$, it is a discrete convex function. However, as N moves from $T-1$ to T, there could even be a sudden drop in value for $\mathcal{H}(N)$, that is, $f_2(T) > f_1(T)$. Thus a binary search of the monotonic difference function $\Delta\mathcal{H}(N) = \mathcal{H}(N) - \mathcal{H}(N-1)$ can be done to find the local minimum on $1 \leq N \leq T-1$, which is then compared to $\mathcal{H}(T)$ to obtain the global minimum for $N \in [1, T]$. The complexity of such an approach is $O\,(log\ N)$.

We conduct numerical experiments to illustrate the results for the one-way trading problem with prepaid opportunities. Let $T = \{10, 20, 30\}$ and $c = \{0, 0.0001, 0.0002, \ldots, 1\}$. N^* and D^{PO} are calculated for different combinations of T and c. The results are shown in Figure 1 and Figure 2.

There are several observations. In Figure 1, for each given T, when the fixed cost c is not greater than a particular threshold $\bar{c}$, N^* is bounded by the length of the trading horizon ($N^* = T$). Meanwhile, $\bar{c}$ is smaller when T is larger. From the numerical results, the value of $\bar{c}$ is about $0.0101, 0.0026, 0.0011$ for the case of $T = 10, 20, 30$ respectively.

In addition, for each T, we find that: (i) N^* is decreasing in c; (ii) D^{PO} is increasing in c; and (iii) the curve of N^* as well as the curve of D^{PO} when $c > \bar{c}$ coincide with the counterparts for a longer trading horizon. When $c > 0.0101$, the curve of N^* (or D^{PO}) for $T = 10$ coincides with that for $T = 20$ and $T = 30$ in Figure 1 (or in Figure 2). When $c > 0.0026$, the curve of N^* (or D^{PO}) for $T = 20$ coincides with that for $T = 30$ in Figure 1 (or in Figure 2). Note that N^* decreases to 1 and D^{PO} increases to 0.5 when $c \geq 0.0556$.

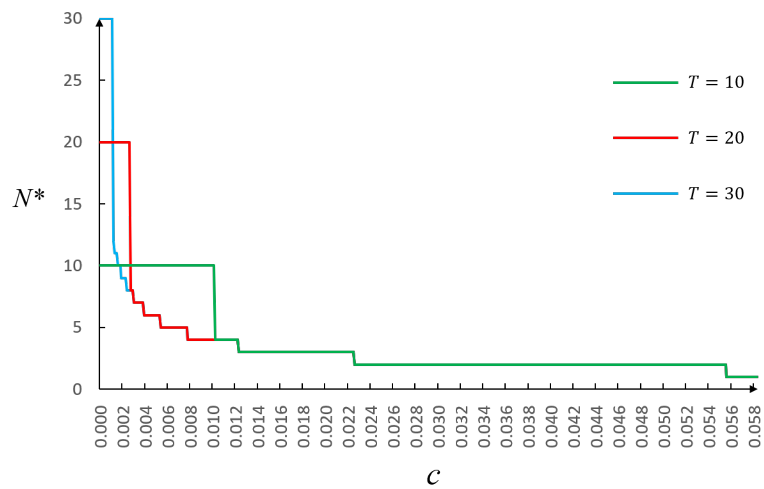

Figure 1. The curve of N^* w.r.t. c when $T = 10, 20, 30$

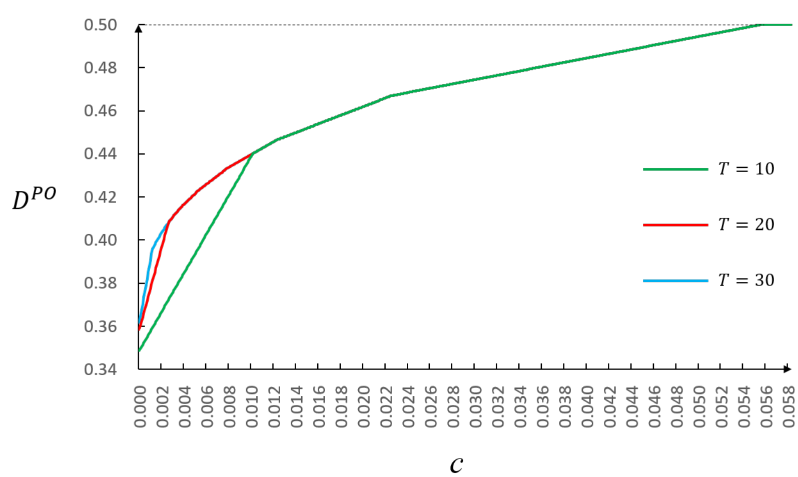

Figure 2. The curve of D^{PO} w.r.t. c when $T = 10, 20, 30$

We also conduct numerical experiments to calculate both the upper bound D^{PO} and the lower bound $D^{FC"}$ on D^{FC} for different fixed costs. Let $c = \{0, 0.001, 0.002, \ldots, 1\}$, $E = 10000$, $F = 1000$, and $T = \{5, 10, 15, \ldots, 100\}$. Figure 3 illustrates the curve of D^{PO} and $D^{FC''}$ with respect to c for the case of $T = \{5, 10, 15\}$. Figure 4 shows the curve of $D^{FC''}/D^{PO}$ with respect to c when $T = \{5, 10, 15, 20, 30\}$. There are several observations from these two figures.

(i) Both D^{PO} and $D^{FC''}$ are increasing in c at a decreasing speed, and each of them reaches the saturation point 0.5 when c is larger than a threshold value. The threshold value for D^{PO} ($\bar{c}^{PO} \approx 0.056$) is smaller than that for $D^{FC''}$ ($\bar{c}^{FC"} \approx 0.086$).

(ii) For an arbitrary c, if $T_1 > T_2$, then $D^{FC''}(T = T_1) \geq D^{FC''}(T = T_2)$ and $D^{PO}(T = T_1) \geq D^{PO}(T = T_2)$. Meanwhile, the curve of $D^{FC''}(T = T_2)$ (or $D^{PO}(T = T_2)$) coincides with that of $D^{FC''}(T = T_1)$ (or $D^{PO}(T = T_1)$) when c exceeds a threshold value.

(iii) $D^{FC''}/D^{PO}$ is greater than 93% for any combination of T and c, which indicates that D^{FC}/D^{PO} is at least 93%. Therefore, the optimal trading policy for problem PO can be used as a heuristic to solve problem FC, and the performance of this heuristic policy is pretty close to that of the optimal policy for problem FC.

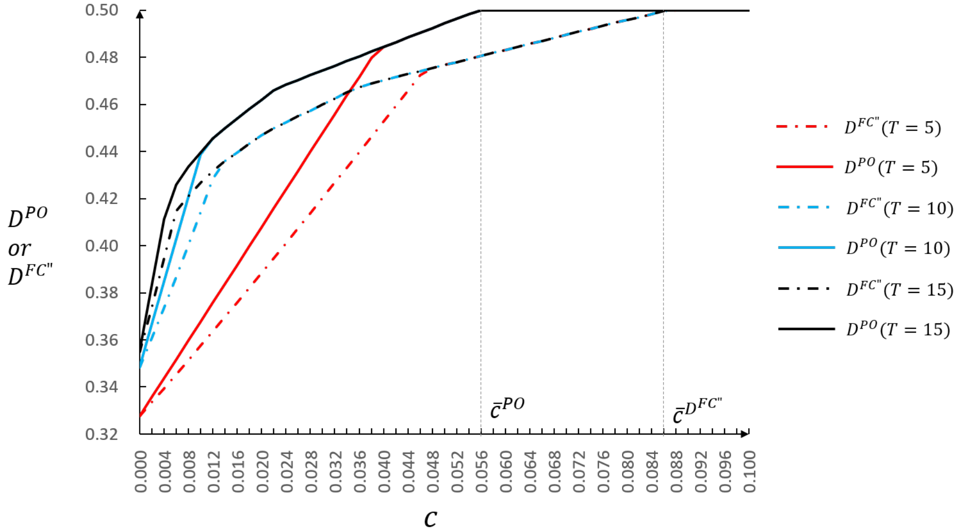

Figure 3. The curve of D^{PO} and $D^{FC"}$ w.r.t. c when $T = 5, 10, 15$

Figure 4. The curve of $D^{FC"}/D^{PO}$ w.r.t. c when $T = 5, 10, 15, 20, 30$

6. Conclusions and Extensions

For the one-way trading problem with fixed costs and limited information about future prices (i.e., price range), we construct the robust optimization model based on CD analysis (called CDA), or the mini-max regret criterion, and pinpoint several important properties about this model. Since it is difficult to solve the original problem directly via backward induction when the trading horizon is relatively long, we try to find out the bounds on it by solving two approximate problems. We first introduce a discrete version of the original problem which determines a lower bound on the CD of the original problem. In order to obtain the upper bound on the CD of the original problem, we transform it to an approximate problem with predetermined number of trading opportunities and propose an online trading policy based on CD analysis (called CDA) to guide the trader's decision process according to actual prices encountered.

There is another purpose of introducing the one-way trading problem with limited opportunities. It connects the well-studied one-way trading problem without opportunity limitations and the time series search problem via a uniform mathematical framework, which facilitates the comparison between these different types of problems, enriches the conclusions of existing literatures, and provides a good foundation for future researches on more complicated one-way trading problems with various practical constraints.

There are several directions to extend this paper. This research can be carried a step further to find analytical solutions to the online inventory control problem with procurement costs and inventory holding cost. In addition, it's worthwhile to relax the assumption of constant price range and consider the situation where the price range in each period depends on the actual price in previous period. For example, the stock price must be within $\pm 10\%$ of last day's price in China stock market.

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Appendix A. Proof of Lemma 1.

Proof of Lemma 1. Let $c' = \frac{c}{M-m}$, $p'_t = \frac{p_t - m}{M-m}$, $k'_t = k_t$, and $s'_{t-1} = \{C'_{t-1}, K'_{t-1}, \bar{p}'_{t-1}\}$ where $C'_{t-1} = \sum_{i=1}^{t-1}\left(p'_i k'_i - c' * 1_{\{k'_i>0\}}\right)$, $K'_{t-1} = \sum_{i=1}^{t-1} k'_i$ and $\bar{p}'_{t-1} = \max(p'_{1:t-1})$. Let $D'_t(s'_{t-1})$ designates the CD of the normalized model in period t with the state s'_{t-1}. Then $c = (M-m) * c'$, $p_t = m + (M-m)p'_t$, $k_t = k'_t$, and

$$\begin{aligned} C_{t-1} &= \sum_{i=1}^{t-1}\left(p_i k_i - c * 1_{\{k_i>0\}}\right) \\ &= \sum_{i=1}^{t-1}\left[(m + (M-m)p'_i)k'_i - (M-m)c' * 1_{\{k'_i>0\}}\right] \\ &= m * K'_{t-1} + (M-m)\sum_{i=1}^{t-1}\left(p'_i k'_i - c' \times 1_{\{k'_i>0\}}\right) \\ &= m * K'_{t-1} + (M-m)C'_{t-1} \end{aligned} \tag{A.1}$$

$$K_{t-1} = \sum_{i=1}^{t-1} k_i = \sum_{i=1}^{t-1} k'_i = K'_{t-1} \tag{A.2}$$

$$\begin{aligned} \bar{p}_{t-1} = \max(p_{1:t-1}) &= \max\left\{m + (M-m)p'_i \ for \ i \in \{1, 2, \ldots, t-1\}\right\} \\ &= m + (M-m)\bar{p}'_{t-1} \end{aligned} \tag{A.3}$$

So,

$$
\begin{aligned}
D_T(s_{T-1}) &= \max_{p_T\in[m,M]} \left\{\bar{p}_T - c - \left(C_{T-1} + p_T(1-K_{T-1}) - c * 1_{\{1-K_{T-1}>0\}}\right)\right\} \\
&= \max_{p'_T\in[0,1]} \{m + (M-m)\bar{p}'_T - (M-m)c' - [m * K'_{T-1} + (M-m)C'_{T-1} \\
&\qquad + (m + (M-m)p'_T)(1-K'_{T-1}) - (M-m)c' * 1_{\{1-K'_{T-1}>0\}}]\} \\
&= (M-m) \max_{p'_T\in[0,1]} \{\bar{p}'_T - c' - [C'_{T-1} + p'_T(1-K'_{T-1}) - c' * 1_{\{1-K'_{T-1}>0\}}]\} \\
&= (M-m) D'_T(s'_{T-1})
\end{aligned} \tag{A.4}
$$

Next, we can prove the following statement: For an arbitrary $t \in \{1, 2, \ldots, T-1\}$, if $D_{t+1}(s_t) = (M-m)D'_{t+1}(s'_t)$, then $D_t(s_{t-1}) = (M-m)D'_t(s'_{t-1})$. The proof is as follows,

$$
\begin{aligned}
D_t(s_{t-1}) &= \max_{p_t\in[m,M]} \min_{k_t\in[0,1-K_{t-1}]} D_{t+1}(s_t) \\
&= \max_{p'_t\in[0,1]} \min_{k'_t\in[0,1-K'_{t-1}]} (M-m)D'_{t+1}(s'_t) = (M-m)D'_t(s'_{t-1})
\end{aligned} \tag{A.5}
$$

Therefore, $D^{FC} = D_1(s_0) = (M-m)D'_1(s'_0) = (M-m)D'^{FC}$.

Appendix B. Proof of Lemma 2.

Proof of Lemma 2. To prove Lemma 2, we only need to show the following statement: for any period $t \in \{1, 2, \ldots, T\}$, C_{t-1} can be separated from the objective function of the subgame in this period, i.e., $D_t(s_{t-1})$. That is, we should verify that $D_t(s_{t-1}) = G_t(K_{t-1}, \bar{p}_{t-1}) - C_{t-1}$ for any $t \in \{1, 2, \ldots, T\}$, where $G_t(K_{t-1}, \bar{p}_{t-1})$ is a function independent of C_{t-1}. The proof consists of the following two steps.

In the first step, we can easily confirm that $D_T(s_{T-1}) = G_T(K_{T-1}, \bar{p}_{T-1}) - C_{T-1}$, where $G_T(K_{T-1}, \bar{p}_{T-1}) = \max_{p_T\in[m,M]} \left\{\max(\bar{p}_{T-1}, p_T) - c - \left(p_T(1-K_{T-1}) - c \times 1_{\{1-K_{T-1}>0\}}\right)\right\}$ since $D_T(s_{T-1})$ is given in (6). In the second step, we will prove the following statement: For an arbitrary $t \in \{1, 2, \ldots, T-1\}$, if $D_{t+1}(s_t) = G_{t+1}(K_t, \bar{p}_t) - C_t$, then

$D_t(s_{t-1}) = G_t(K_{t-1}, \bar{p}_{t-1}) - C_{t-1}$. The proof is straightforward,

$$\begin{aligned} D_t(s_{t-1}) &= \max_{p_t \in [m,M]} \min_{k_t \in [0, 1-K_{t-1}]} D_{t+1}(s_t) \\ &= \max_{p_t \in [m,M]} \min_{k_t \in [0, 1-K_{t-1}]} [G_{t+1}(K_t, \bar{p}_t) - C_t] \\ &= \max_{p_t \in [m,M]} \min_{k_t \in [0, 1-K_{t-1}]} [G_{t+1}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) - (C_{t-1} + p_t k_t - c \times 1_{\{k_t > 0\}})] \\ &= \{ \max_{p_t \in [m,M]} \min_{k_t \in [0, 1-K_{t-1}]} [G_{t+1}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) - (p_t k_t - c \times 1_{\{k_t > 0\}})]\} - C_{t-1}. \end{aligned} \tag{B.1}$$

Thus, $D_t(s_{t-1}) = G_t(K_{t-1}, \bar{p}_{t-1}) - C_{t-1}$, where $G_t(K_{t-1}, \bar{p}_{t-1}) = \max_{p_t \in [m,M]} \min_{k_t \in [0,1-K_{t-1}]} [G_{t+1}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) - (p_t k_t - c \times 1_{\{k_t > 0\}})]$.

Appendix C. Proof of Lemma 3.

Proof of Lemma 3. The proof of Lemma 3 consists of two steps. In the first step, we can confirm that $B_T(K_{T-1}, \bar{p}_{T-1}) = max\,(K_{T-1}, \bar{p}_{T-1}) - c + c \times 1_{\{1-K_{T-1}>0\}}$ is weakly increasing in K_{T-1} when $K_{T-1} < 1$. In the second step, we prove the following statement: If $B_{t+1}(K_t, \bar{p}_t)$ is weakly increasing in K_t when $K_t < 1$, then $B_t(K_{t-1}, \bar{p}_{t-1})$ is weakly increasing in K_{t-1} when $K_{t-1} < 1$. In other words, we need to prove that $B_t(K_{t-1} + \delta, \bar{p}_{t-1}) \geq B_t(K_{t-1}, \bar{p}_{t-1})$ when $K_{t-1} < 1$ for an arbitrary $\delta \in [0, 1 - K_{t-1})$.

For an arbitrary $\delta \in [0, 1 - K_{t-1})$,

$$B_t(K_{t-1} + \delta, \bar{p}_{t-1}) = \max_{p_t \in [0,1]} \min_{k_t \in [0, 1-K_{t-1}-\delta]} B_{t+1}(K_{t-1} + \delta + k_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t>0\}}) \tag{C.1}$$

Since $B_{t+1}(K_t, \bar{p}_t)$ is weakly increasing in K_t when $K_t < 1$, we have $B_{t+1}(K_{t-1} + \delta +$

$k_t, \bar{p}_t) \geq B_{t+1}(K_{t-1} + k_t, \bar{p}_t)$. So,

$$\begin{aligned} &\min_{k_t \in [0, 1-K_{t-1}-\delta)} B_{t+1}(K_{t-1} + \delta + k_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}) \\ &\geq \min_{k_t \in [0, 1-K_{t-1}-\delta)} B_{t+1}(K_{t-1} + k_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}) \\ &\geq \min_{k_t \in [0, 1-K_{t-1})} B_{t+1}(K_{t-1} + k_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}). \end{aligned} \tag{C.2}$$

In addition, when $k_t = 1 - K_{t-1} - \delta > 0$ in the minimization problem of (C.1), we have

$$B_{t+1}(1, \bar{p}_t) - (p_t(1 - K_{t-1} - \delta) - c) \geq B_{t+1}(1, \bar{p}_t) - (p_t(1 - K_{t-1}) - c \times 1_{\{1-K_{t-1} > 0\}}) \tag{C.3}$$

Therefore,

$$\begin{aligned} &\min_{k_t \in [0, 1-K_{t-1}-\delta]} B_{t+1}(K_{t-1} + \delta + k_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}) \\ &\quad \geq \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_{t-1} + k_t, \bar{p}_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}) \end{aligned} \tag{C.4}$$

Thus $B_t(K_{t-1} + \delta, \bar{p}_{t-1}) \geq B_t(K_{t-1}, \bar{p}_{t-1})$ holds.

Appendix D. Proof of Lemma 4.

In any period t, if the adversary sets the current price at $p_t = m$, then the trader faces the following problem according to (8),

$$\min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_{t-1}) - (0 - c * 1_{\{k_t > 0\}}) \tag{D.1}$$

Since $B_{t+1}(K_t, \bar{p}_{t-1})$ is increasing in K_t (according to Lemma 3), the minimum of it is obtained at $k_t = 0$. Thus Lemma 4 is proved.

Appendix E. Proof of Lemma 5.

Lemma 5 is consist of the following two statements.

Statement 1. In an arbitrary period t, if the adversary chooses to reduce the price in this period, then it is optimal for her to reduce it to the bottom price 0.

Statement 2. If the adversary has already reduced the price to 0 in a certain period, then it is optimal for her to keep the price at 0 in the subsequent period. In other words, given the historical state K_{t-1} and $\bar{p}_{t-1}$, if it is optimal for the adversary to set $p_t = 0$ in period t, then it is no worse for her to set $p_{t+1} = 0$ than to set p_{t+1} at other values within $(0, 1]$.

We prove Statement 1 first. For any $p_t \in [0, \bar{p}_{t-1}]$, there is

$$\begin{aligned} B_{t+1}(K_t, \bar{p}_t) - (p_t k_t - c * 1_{\{k_t>0\}}) &= B_{t+1}(K_t, \bar{p}_{t-1}) - (p_t k_t - c * 1_{\{k_t>0\}}) \\ &\leq B_{t+1}(K_t, \bar{p}_{t-1}) - (0 - c * 1_{\{k_t>0\}}) \end{aligned} \tag{E.1}$$

so,

$$\begin{aligned} &\min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_t) - (p_t k_t - c * 1_{\{k_t>0\}}) \\ &\qquad \leq \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_{t-1}) - (0 - c * 1_{\{k_t>0\}}) \end{aligned} \tag{E.2}$$

Note that the right-hand-side of the inequality (E.2) is independent of p_t, therefore,

$$\begin{aligned} &\max_{p_t \in [0, \bar{p}_{t-1}]} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_t) - (p_t k_t - c * 1_{\{k_t>0\}}) \\ &\leq \max_{p_t \in [0, \bar{p}_{t-1}]} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_{t-1}) - (0 - c * 1_{\{k_t>0\}}) \\ &= \max_{p_t \in \{0\}} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_t) - (p_t k_t - c * 1_{\{k_t>0\}}) \end{aligned} \tag{E.3}$$

Hence Statement 1 is proved.

Next, we prove Statement 2. We first prove the following preparatory statement (denoted by Statement 3) : for any $t \in \{1, 2, \ldots, T-1\}$, $B_t(K_{t-1}, \bar{p}_{t-1}) \geq B_{t+1}(K_{t-1}, \bar{p}_{t-1})$, and the equality holds when $p_t = 0$.

$$B_t(K_{t-1}, \bar{p}_{t-1}) = \max_{p_t \in [0,1]} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) - (p_t k_t - c * 1_{\{k_t > 0\}}) \quad \text{(E.4)}$$

If p_t is set at 0 on the right-hand-side, then according to Lemma 4, we can conclude that $k_t = 0$, and $B_t(K_{t-1}, \bar{p}_{t-1}) = B_{t+1}(K_{t-1}, \bar{p}_t)$. Therefore, Statement 3 is proved.

Now we are ready to prove Statement 2. Since it is optimal for the adversary to set $p_t = 0$ in period t, we know that

$$\max_{p_t \in [0,1]} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_t, \bar{p}_t) - (p_t k_t - c * 1_{\{k_t > 0\}}) = B_{t+1}(K_{t-1}, \bar{p}_{t-1}) \quad \text{(E.5)}$$

We can also conclude that $k_t = 0$ according to Lemma 4, thus the state variables in the beginning of period $t+1$ are $K_t = K_{t-1}$ and $\bar{p}_t = \bar{p}_{t-1}$. Therefore,

$$\begin{aligned}
&\max_{p_{t+1} \in [0,1]} \min_{k_{t+1} \in [0, 1-K_{t-1}]} B_{t+2}(K_{t-1} + k_{t+1}, max(\bar{p}_{t-1}, p_{t+1})) - (p_{t+1} k_{t+1} - c * 1_{\{k_{t+1} > 0\}}) \\
&= \max_{p_t \in [0,1]} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+2}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) - (p_t k_t - c * 1_{\{k_t > 0\}}) \\
&\leq \max_{p_t \in [0,1]} \min_{k_t \in [0, 1-K_{t-1}]} B_{t+1}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) - (p_t k_t - c * 1_{\{k_t > 0\}}) \\
&= B_{t+1}(K_{t-1}, \bar{p}_{t-1}) \\
&= B_{t+2}(K_{t-1}, \bar{p}_{t-1})
\end{aligned} \quad \text{(E.6)}$$

The inequality in the third line of (E.6) holds because $B_{t+2}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t)) \leq B_{t+1}(K_{t-1} + k_t, max(\bar{p}_{t-1}, p_t))$ according to Statement 3. The equality in the forth line of (E.6) holds based on (E.5). The equality in the last line of (E.6) holds because of Statement 3 in the case of $p_{t+1} = 0$. Statement 2 is thus proved.

Appendix F. Proof of Theorem 1.

Proof of Theorem 1. Theorem 1 can be easily derived from Lemmas 4 and 5.

Appendix G. Proof of Lemma 6.

Proof of Lemma 6. We can first confirm that $\breve{B}_T(K_{T-1}, p_{T-1}) = \max(K_{T-1}, p_{T-1}) - c + c \times 1_{\{1-K_{T-1}>0\}}$ is weakly increasing in K_{T-1} when $K_{T-1} < 1$. Then we prove the following statement: For any $t \in \{1, 2, \ldots, T-1\}$, if $\breve{B}_{t+1}(K_t, p_t)$ is weakly increasing in K_t when $K_t < 1$, then $\breve{B}_t(K_{t-1}, p_{t-1})$ is weakly increasing in K_{t-1} when $K_{t-1} < 1$. In other words, we only need to prove that $\breve{B}_t(K_{t-1} + \delta, p_{t-1}) \geq \breve{B}_t(K_{t-1}, p_{t-1})$ when $K_{t-1} < 1$ for an arbitrary $\delta \in [0, 1 - K_{t-1})$.

According to (11), for an arbitrary $\delta \in [0, 1 - K_{t-1})$,

$$\breve{B}_t(K_{t-1} + \delta, p_{t-1}) = \max\{p_{t-1}, \max_{p_t \in [p_{t-1}, 1]} \breve{A}_t(K_{t-1} + \delta, p_{t-1}|p_t)\} \tag{G.1}$$

where

$$\breve{A}_t(K_{t-1} + \delta, p_{t-1}|p_t) = \min_{k_t \in [0, 1-K_{t-1}-\delta]} \breve{B}_{t+1}(K_{t-1} + \delta + k_t, p_t) - (p_t k_t - c \times 1_{\{k_t>0\}}) \tag{G.2}$$

Since $\breve{B}_{t+1}(K_t, p_t)$ is weakly increasing in K_t when $K_t < 1$, we have $\breve{B}_{t+1}(K_{t-1} + \delta + k_t, p_t) \geq \breve{B}_{t+1}(K_{t-1} + k_t, p_t)$ when $k_t < 1 - K_{t-1} - \delta$. So

$$\begin{aligned}
&\min_{k_t \in [0, 1-K_{t-1}-\delta)} \breve{B}_{t+1}(K_{t-1} + \delta + k_t, p_t) - (p_t k_t - c \times 1_{\{k_t>0\}})\} \\
\geq &\min_{k_t \in [0, 1-K_{t-1}-\delta)} \breve{B}_{t+1}(K_{t-1} + k_t, p_t) - (p_t k_t - c \times 1_{\{k_t>0\}}) \\
\geq &\min_{k_t \in [0, 1-K_{t-1})} \breve{B}_{t+1}(K_{t-1} + k_t, p_t) - (p_t k_t - c \times 1_{\{k_t>0\}})
\end{aligned} \tag{G.3}$$

In addition, when $k_t = 1 - K_{t-1} - \delta > 0$ in the minimization problem of (G.2), we have

$$\breve{B}_{t+1}(1, p_t) - (p_t(1 - K_{t-1} - \delta) - c) \geq \breve{B}_{t+1}(1, p_t) - (p_t(1 - K_{t-1}) - c \times 1_{\{1-K_{t-1}>0\}}) \tag{G.4}$$

So we have $\breve{A}_t(K_{t-1}+\delta, p_{t-1}|p_t) \geq \breve{A}_t(K_{t-1}, p_{t-1}|p_t)$, thus $\breve{B}_t(K_{t-1}+\delta, p_{t-1}) \geq \breve{B}_t(K_{t-1}, p_{t-1})$ holds.

Appendix H. Proof of Theorem 2.

Proof of Theorem 2. For the two-period problem, in the second period, the trader's choice is to sell out all the remaining goods (i.e., $k_2 = 1 - k_1$). The adversary's problem in this period is

$$B_2(k_1, p_1) = \max_{p_2 \in [0,1]} \left[max(p_1, p_2) - c - \left(p_2(1 - k_1) - c * 1_{\{1-k_1>0\}} \right) \right] \tag{H.1}$$

according to (9). Since the objective function of the outer maximization problem is decreasing in p_2 when $p_2 <= p_1$ and is increasing in p_2 when $p_2 > p_1$, we can conclude that the maximum is achieved at $p_2 = 0$ or $p_2 = 1$, which resulting in

$$B_2(k_1, p_1) = max(p_1, k_1) - c + c * 1_{\{1-k_1>0\}} \tag{H.2}$$

According to (8), the subgame between the adversary and the trader in the first period is as following,

$$B_1(0, 0) = \max_{p_1 \in [0,1]} \min_{k_1 \in [0,1]} B_2(k_1, p_1) - (p_1 k_1 - c * 1_{\{k_1>0\}}) \tag{H.3}$$

We first solve the inner minimization problem in (H.3) for the trader. The objective function

$$B_2(k_1, p_1) - (p_1 k_1 - c * 1_{\{k_1>0\}}) = \begin{cases} p_1 & if\ k_1 = 0 \\ max(p_1, k_1) - p_1 k_1 + c & if\ k_1 \in (0, 1) \\ 1 - p_1 & if k_1 = 1 \end{cases} \tag{H.4}$$

In the interval $k_1 \in (0, 1)$, the objective function is decreasing in k_1 when $k_1 <= p_1$ and is increasing in k_1 when $k_1 > p_1$, so there is a local minimum $\frac{1}{4} + c - (p_2 - \frac{1}{2})^2$ obtained at $k_1^* = p_1$. Comparing the objective value at $k_1 = 0$, $k_1 = 1$ and $k_1 = k_1^* = p_1$, we obtain the optimal trading volume for the trader in the first period, i.e., $\hat{k}_1(p_1)$, and the optimal objective value is

$$\hat{B}_1(p_1) = \begin{cases} p_1 & if\ p_1 \leq p_{11} \\ \frac{1}{4} + c - (p_2 - \frac{1}{2})^2 & if\ p_{11} < p_1 < p_{12} \\ 1 - p_1 & if\ p_1 \geq p_{12} \end{cases} \tag{H.5}$$

Now we solve the outer maximization problem in (H.3) for the adversary, which is $\max_{p_1 \in [0,1]} \hat{B}_1(p_1)$. We obtain $p_1^* = \frac{1}{2}$, and the minimal CD for the two-period problem is $D^{FC} = \min(1/4 + c, 1/2)$.

Appendix I. Proof of Lemma 7.

Proof of Lemma 7. Given x, suppose $y^*(x) = \arg\min_{y \in Y} f_1(x, y)$ and $y'^*(x) = \arg\min_{y \in Y'} f_2(x, y)$. Since $Y' \subseteq Y$, $y'^*(x)$ is a feasible but not necessarily optimal solution of the minimization problem $\min_{y \in Y}\ f_1(x, y)$. In other words, $\min_{y \in Y} f_1(x, y) \leq f_1(x, y'^*(x))$. Therefore,

$$\min_{y \in Y}\ f_1(x, y) = f_1(x, y^*(x)) \leq f_1(x, y'^*(x)) \leq f_2(x, y'^*(x)) = \min_{y \in Y'}\ f_2(x, y) \tag{I.1}$$

Next, we go on to prove $\max_{x \in X'} f_1(x, y^*(x)) \leq \max_{x \in X} f_2(x, y'^*(x))$. Suppose $x'^* = \arg\max_{x \in X'} f_1(x, y^*(x))$. Since $X' \subseteq X$, x'^* is a feasible but not necessarily optimal solution of the maximization problem $\max_{x \in X} f_2(x, y'^*(x))$. In other words, $f_2(x'^*, y'^*(x'^*)) \leq \max_{x \in X} f_2(x, y'^*(x))$. Therefore,

$$\max_{x \in X'} f_1(x, y^*(x)) = f_1(x'^*, y^*(x'^*)) \leq f_2(x'^*, y'^*(x'^*)) \leq \max_{x \in X} f_2(x, y'^*(x)) \tag{I.2}$$

Appendix J. Proof of Theorem 3.

Proof of Theorem 3. Since the discretization of the adversary's decision just reduces her own strategy space when maximizing the CD, it can still guarantee the lower bound according to Corollary 1. Now we prove that, if $K_{t-1} = i \times \Delta_k$ and $p_t = j \times \Delta_p \geq p_{t-1}$, there is

$$\breve{A}_t(K_{t-1}, p_{t-1}|p_t) \geq \dot{A}_t(i,j) \tag{J.1}$$

where

$$\breve{A}_t(K_{t-1}, p_{t-1}|p_t) = \min_{k_t \in [0, 1-K_{t-1}]} \breve{B}_{t+1}(K_{t-1} + k_t, p_t) - (p_t k_t - c \times 1_{\{k_t > 0\}}) \tag{J.2}$$

$$\dot{A}_t(i,j) = min\left\{\dot{B}_{t+1}(i,j),\ \ \mathcal{G}_t(i,j)\right\} - j \times \Delta_p \Delta_k \tag{J.3}$$

with $\mathcal{G}_t(i,j) = \min\left\{\dot{B}_{t+1}(i+e,j) - j\Delta_p \times e\Delta_k + c \ for \ e = 1, 2, \ldots, E-i\right\}$.

Suppose $K_{t-1} = i \times \Delta_k$ and $p_t = j \times \Delta_p$. For an arbitrary $k_t \in (e\Delta_k, (e+1)\Delta_k)$, $e \in \{0, 1, \ldots, E-i-1\}$, let $\mathring{k} = k_t - e\Delta_k < \Delta_k$, there is

$$\begin{aligned}
&\min_{k_t \in (e\Delta_k, (e+1)\Delta_k)} \breve{B}_{t+1}(K_{t-1} + k_t, p_t) - p_t k_t + c \\
&\geq \min_{k_t \in (e\Delta_k, (e+1)\Delta_k)} \breve{B}_{t+1}(K_{t-1} + k_t - \mathring{k}, p_t) - p_t k_t + c \\
&= \min_{k_t \in (e\Delta_k, (e+1)\Delta_k)} \breve{B}_{t+1}(K_{t-1} + k_t - \mathring{k}, p_t) - p_t(k_t - \mathring{k} + \mathring{k}) + c \\
&\geq \min_{k_t \in (e\Delta_k, (e+1)\Delta_k)} \breve{B}_{t+1}(K_{t-1} + k_t - \mathring{k}, p_t) - p_t(k_t - \mathring{k} + \Delta_k) + c \\
&= \breve{B}_{t+1}(K_{t-1} + e \times \Delta_k, p_t) - p_t(e \times \Delta_k + \Delta_k) + c \\
&= \dot{B}_{t+1}(i+e, j) - j\Delta_p \times (e+1)\Delta_k + c
\end{aligned} \tag{J.4}$$

The first inequality in (J.4) is based on Lemma 6, while the second inequality in (J.4) holds because $\mathring{k} \leq \Delta_k$. Meanwhile, for any $e \in \{1, 2, \ldots, E-i\}$, there is

$$\begin{aligned}\breve{B}_{t+1}(K_{t-1} + e\Delta_k, p_t) - p_t \times e\Delta_k + c &= \dot{B}_{t+1}(i+e, j) - j\Delta_p \times e\Delta_k + c \\ &\geq \dot{B}_{t+1}(i+e, j) - j\Delta_p \times (e+1)\Delta_k + c\end{aligned} \tag{J.5}$$

Therefore,

$$\begin{aligned}&\min_{k_t \in (0, 1-K_{t-1}]} \breve{B}_{t+1}(K_{t-1} + k_t, p_t) - p_t k_t + c \\ &\geq \min \left\{ \dot{B}_{t+1}(i+e, j) - j\Delta_p \times (e+1)\Delta_k + c \;\; for\ e \in \{0, 1, \ldots, E-i\} \right\} \\ &= \min \left\{ \dot{B}_{t+1}(i+e, j) - j\Delta_p \times e\Delta_k + c \;\; for\ e \in \{0, 1, \ldots, E-i\} \right\} - j\Delta_p\Delta_k \\ &= min \left\{ \dot{B}_{t+1}(i, j) + c, \;\; \mathcal{G}_t(i, j) \right\} - j\Delta_p\Delta_k\end{aligned} \tag{J.6}$$

Finally, we check the special point $k_t = 0$ in (J.1). Since $\breve{B}_{t+1}(K_{t-1}, p_t) = \dot{B}_{t+1}(i, j)$, we have $\breve{B}_{t+1}(K_{t-1}, p_t) \geq \dot{B}_{t+1}(i, j) - j\Delta_p\Delta_k$. Thus

$$\begin{aligned}\breve{A}_t(K_{t-1}, p_{t-1}|p_t) &= \min_{k_t \in [0, 1-K_{t-1}]} \breve{B}_{t+1}(K_{t-1} + k_t, p_t) - p_t k_t + c \\ &\geq min \left\{ \dot{B}_{t+1}(i, j), \;\; \dot{B}_{t+1}(i, j) + c, \;\; \mathcal{G}_t(i, j) \right\} - j\Delta_p\Delta_k \\ &= min \left\{ \dot{B}_{t+1}(i, j), \;\; \mathcal{G}_t(i, j) \right\} - j\Delta_p\Delta_k \\ &= \dot{A}_t(i, j)\end{aligned} \tag{J.7}$$

Appendix K. Proof of Theorem 4.

Proof of Theorem 4. We analyze the calculation complexity of problem FC” in the backward induction process. In the last period T, we only need to calculate each cell of the matrix $\dot{B}_T(i, j)$ according to (20). The calculation volume is $(E+1) * (F+1)$.

In any period $t \in \{1, 2, \ldots, T-1\}$, we need to calculate both the matrices $\dot{A}_t$ and $\dot{B}_t$. When calculating each cell in the ith row of $\dot{A}_t$, we need to consider all the possible values of k_t, including $\{e\Delta_k \; for\ e = 0, 1, \ldots, E-i\}$. So, the calculation volume for each column of the matrix $\dot{A}_t$ is $(F+1)\sum_{i=0}^{E}(E-i+1) = (E+1)(E+2)/2$. Fortunately, according to (23), this calculation volume can be reduced to $(E+1)$. Therefore, the calculation volume of computing the matrix $\dot{A}_t$ is $(E+1)*(F+1)$.

When calculating each cell in the jth column of $\dot{B}_t$, we need to consider all the possible values of p_t, including $\{j*\Delta_p + f\Delta_p for\ f = 0, 1, \ldots, F-j+1\}$. So, the calculation volume for each row of the matrix $\dot{B}_t$ is $(E+1)\sum_{j=0}^{F}(F-j+2) = (F+2)(F+3)/2-1$. Fortunately, according to (22), this calculation volume can be reduced to $(F+1)$. Therefore, the calculation volume of computing the matrix $\dot{B}_t$ is $(E+1)*(F+1)$.

Therefore, the calculation complexity of problem FC" is $O(T*E*F)$.

Appendix L. Proof of Theorem 5.

Proof of Theorem 5. According to the definition of Problem FC-LO, its objective function is the same as that of Problem FC, but the strategy space for the trader in this problem is much reduced compared with the strategy space for the trader in Problem FC. Therefore, $D^{FC} \leq D^{FC-LO}$ based on Corollary 1.

Next we prove $D^{FC-LO} \leq D^{PO}$. Assume that N is the number of prepaid opportunities in Problem FC-LO, and N is also the maximum number of transactions allowed in Problem FC-LO. The game between the trader and the adversary in both Problem FC-LO and Problem PO can be described by the same model in (5), and the strategy spaces of both players are the same in problem $FC-LO$ and problem PO. The only difference between the two problems lies in their objective functions, or the boundary condition $D_{T+1}(p_{1:T}, k_{1:T})$. The objective function of Problem PO is

$$D_{T+1}(p_{1:T}, k_{1:T}) = \underbrace{[max(p_{1:T}) - c]}_{Offline\ Profit} - \underbrace{\left[\sum_{t=1}^{T} p_t k_t - c \times N\right]}_{Online\ Profit} \tag{L.1}$$

while the objective function of Problem FC-LO with at most N opportunities is

$$\hat{D}_{T+1}(p_{1:T}, k_{1:T}) \quad = \quad \underbrace{[max(p_{1:T}) - c]}_{Offline\ Profit} - \underbrace{\left[\sum_{t=1}^{T}(p_t k_t - c \times 1_{\{k_t>0\}})\right]}_{Online\ Profit} \tag{L.2}$$

$D_{T+1}(p_{1:T}, k_{1:T})$ is obviously not smaller than $\hat{D}_{T+1}(p_{1:T}, k_{1:T})$, since the former deducts $N{*}c$ while the later only deducts the actual fixed costs $c{\times}\sum_{t=1}^{T} 1_{\{k_t>0\}} <= N{*}c$. According to Corollary 1, $D^{FC-LO} \leq D^{PO}$.