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Keywords: regime shift, risks model, U.S. stock, treasury bond market.
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1 Introduction

Stocks and nominal bonds are two primary asset classes on investors' portfolio menus. It is very important to have a general equilibrium model to provide a coherent explanation of the risks and returns of these two markets simultaneously. The absence of arbitrage opportunities implies that cross-market restrictions should be respected in any such models. Campbell, Sunderam, and Viceira (2017) document the empirical evidence on stochastic correlation between stock and bond returns. Figure 1 plots the time-varying correlation between stock and long-term government bond returns, which is calculated based on a 3-year centered moving window of monthly real returns of stock and bond. As Figure 1 shows, the correlation displays tremendous fluctuations, and also occasionally switches sign. Specifically, the correlation is usually positive; however, in periods like the Great Depression of the 1930s and the more recent global financial crisis, Treasury bonds performed well as hedges for stock returns. Based on CAPM, these movements are significant enough to cause substantial changes (even switching signs) in the risk premium on Treasury bonds. This has important implications for investors since the risks for nominal bonds are changing, rather than being constant as often assumed in traditional portfolio choice theory. Despite tremendous progress in the general equilibrium to model risks and returns for bond or stock markets separately, very few take into account the joint behaviors of these two asset classes.

In this paper, we study the joint determinants of stock and bond returns in a Bansal and Yaron (2004) type of long-run risks framework. The long-run risks framework features a recursive preference for an early resolution of uncertainty, low frequency movements in both expected consumption and expected inflation, and time-varying consumption and inflation volatilities. Beyond these, we allow for an additional novel feature: regime shifts in consumption and inflation dynamics; in particular, the means, volatilities, and correlation structure of consumption and inflation dynamics are regime-dependent.

This rational expectations general equilibrium framework can (1) jointly match the dynamics of consumption, inflation and aggregate cash flow; (2) generate time-varying and switching signs of stock and bond correlations, as well as generate switching signs of bond risk premium; and (3) quantitatively reproduce another long list of salient empirical features in stock and bond markets. Specifically, our model is able to reproduce a high equity premium of 8.34% and an upward sloping unconditional nominal term structure with around 0.6% of a 5-1 year term spread, which are both consistent with the data. In addition, our model also generates significant time-variation in equity and bond return premia as well as regime shifts in real and nominal yield curves across business cycles. When we run the bond return predictability regression of Cochrane and Piazzesi (2005) inside our model, we find that the single factor projection captures 13% of the variation in bond returns, which is close to the data counter-part.

In this paper, we broadly classifies the economy into three regimes: expansion, contraction, and deep-recession. The expansion regime features a high consumption growth, a medium level of inflation, and low uncertainty (which is measured by consumption and inflation volatilities). In the contraction regime, the growth rate is lower, uncertainty is higher, and the inflation level is also higher. One might consider this regime to be a stagflation regime, in which low growth and high inflation coexist. A typical sample episode occurred in the late 1970s and early 1980s. The deep-recession regime features the lowest growth level and the highest level of uncertainty. As opposed to regular contraction, this regime has very low inflation, since deflation rather than inflation is more of a concern at this time. A key ingredient that is different across these three regimes is the **nominal-real correlation**, which refers specifically to the correlation between shocks to expected growth and inflation factors. In the first two regimes, positive news to expected inflation factor indicates lower future expected growth; however, in the deep recession regime, the relationship is just the opposite. In other words, positive news to expected inflation indicates higher future expected growth. We provide empirical evidence to support this channel in the next section. This ingredient is very important for generating tremendous movements (and potentially switching signs) in nominal bond risk premium as well as stock-bond correlation.

We use a regime switching dynamic correlation ($RSDC$) model by Pelletier (2006) to specify the correlation structure between expected consumption and inflation shocks; in particular, the correlation is constant within each regime, but becomes different and even switches sign across different regimes (i.e., switching signs of nominal-real correlations). This setup leads to sign-switching market prices of long-run inflation risks, the magnitude of which are magnified by high persistence of expected growth and inflation factor. This feature, therefore, quantitatively generates sign-switching nominal bond risk premium. In the meantime, this correlation structure also generates time-varying and sign-switching stock and bond correlations, consistent with empirical evidence.

Beyond this regime-specific correlation structure, we also allow for the mean levels of consumption growth and inflation to be different across regimes. In the equilibrium, the mean level acts as a "level" factor, driving regime-shifts in levels of both real and nominal yield curves, consistent with the findings of Bansal and Zhou (2002), which features a reduced form statistical model with regime switching.

Our model can also potentially rationalize the sizable variance risk premium and its predictability of short-run stock returns, as documented by Bollerslev, Tauchen, and Zhou (2009) and Drechsler and Yaron (2011). In this model, the regime shift risk is priced, as in the reduced form regime switching term structure model in Dai, Singleton, and Yang (2007). The regime shift risk premium, caused by different means, volatilities, and correlations across regimes, determines the discrepancy between statistical and risk-neutral transition probabilities, and therefore potentially leads to a sizable variance risk premium. Since both the variance risk premium and equity premium are driven by time-varying macroeconomic uncertainty, the variance risk premium is able to predict stock returns. This is a future direction to be explored.

The works closest to our paper include Hasseltoft and Burkhardt (2012), David and Veronesi (2013), Song (2017) and Campbell, Pflueger, and Viceira (2020). Hasseltoft and Burkhardt (2012) find an inverse relation between stock-bond correlations and correlations of growth and inflation.

They rationalize their findings in a consumption-based asset pricing model with regime switching. David and Veronesi (2013) estimate a general equilibrium model in which agents learn about composite economic and inflation regimes. They show that variations in investors' beliefs about inflation regimes strongly affect the signs of stock-bond correlations. Song (2017) estimates a model that allows for shifts in the aggressiveness of monetary policy to account for the sign-switching stock-bond correlations. Finally, Campbell, Pflueger, and Viceira (2020) study a consumption-based New Keynesian model with habit; they estimate that the correlation between inflation and the output gap switched from negative to positive in 2001, which explains the sign-switching correlations.

Our paper is distinct from these papers in several ways. First, we extend our sample period back to 1926 and find that, in addition to the post-2001 period, another two important episodes – the Great Depression of the 1930s and the recessions around 1960 – also exhibit negative stock-bond correlations. Therefore, we argue that the sign-switching stock-bond correlations cannot be fully attributed to different monetary policies but should rather be an inherent feature of business cycles. Second, we emphasize regime-switching correlations between long-run expected consumption growth and inflation. We provide empirical evidence using Survey of Professional Forecasters data to support this channel. Third, our model calibration not only generates sign-switching stock-bond correlations, but also quantitatively accounts for other salient features in stock and bond market, including time-varying equity and bond premia, regime shifts in bond yield curves, and the violation of expectations hypothesis of bond returns. We model the macroeconomic volatility as a autoregressive Gamma process, with regime specific mean and volatility levels. This channel is very important for generating significant time-varying bond risk premium, and can quantitatively reproduce the violations of expectations hypothesis as well as the Cochrane and Piazzesi (2005) single-factor regressions.

The rest of the paper is organized as follows. In Section 2, we document the empirical evidence on changing inflation risks and nominal-real correlations. In Section 3, we present a long-run risks model with regime shifts. In Section 4, we present the solution to our model and discuss its theoretical implications. In Section 5, we calibrate our model and discuss its implications for bond and stock markets. We then conclude our paper with Section 6.

2 Empirical Evidence

In this section, we provide more empirical evidence that motivates the "nominal-real correlation" channel. Figure 2 summarizes the CAPM beta of inflation, which captures the comovement of inflation shocks with stock returns. We estimate a VAR(1) model for inflation, stock returns (real), and the three-month Treasury bill returns over a centered rolling window of 5-years' quarterly data, and then compute the CAPM beta of inflation. Figure 2 shows that the beta of realized inflation moves tremendously and occasionally switches sign. When comparing Figure 2 to Figure 1, one can find that the periods of positive CAPM beta of inflation line up quite well with a negative stock-bond relationship. This is intuitive since high inflation is associated with high bond yields and low bond returns. This clearly implies that the time-varying and switching signs of stock-bond correlations are closely related to changing inflation risks.

Figure 3 provides direct evidence of the stochastic nature of nominal-real correlations. We follow a similar moving window quarterly VAR approach to compute the industrial production growth beta of inflation over the long sample (first panel), as well as the consumption beta of inflation over the post-war sample (second panel). The latter panel uses GDP and inflation expectations from the Survey of Professional Forecasters data for the period $1968Q3 - 2021Q4$ to proxy for the correlation of expected growth and inflation factors. All three figures display quite similar patterns, and show that the nominal-real correlation does move significantly, which is an important channel to pursue.

3 A Long-run Risks Model with Regime Shifts

3.1 Preferences

We consider a discrete-time endowment economy. The investors' preferences over the uncertain consumption stream C_t can be described by the Kreps-Porteus, Epstein-Zin recursive utility function (see Epstein and Zin (1989); Kreps and Porteus (1978)):

$$U_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\left(E_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \tag{1}$$

in which $\delta \in (0,1)$ is the time discount factor, γ is the risk aversion parameter, and ψ is the intertemporal elasticity of substitution (IES). Parameter θ is defined by $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Its sign is determined by the magnitudes of the risk aversion and the elasticity of substitution, so that if $\psi > 1$ and $\gamma > 1$, then θ will be negative. Note that when $\theta = 1$ (i.e., $\gamma = \frac{1}{\psi}$), this recursive preference collapses to the standard expected utility. As is pointed out by Epstein and Zin (1989), the agent in this case is indifferent to when the uncertainty of the consumption path is resolved. When risk aversion exceeds (is less than) the reciprocal of IES, the agent prefers early (late) resolution of uncertainty of consumption path. In the long-run risks model, agents prefer early resolution of uncertainty.

The logarithm of the Intertemporal Marginal Rate of Substitution (IMRS) is given by:

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1)\, r_{c,t+1}, \tag{2}$$

in which $\Delta c_{t+1} = log(C_{t+1}/C_t)$ is the log growth rate of aggregate consumption and $r_{c,t+1}$ is the log of the return (i.e., continuous return) on an asset that delivers aggregate consumption as its dividends for each time period. This return is not observed in the data. Rather, it is different from the observed return on the market portfolio as the levels of market dividends and consumption are not equal: aggregate consumption is much larger than aggregate dividends. Therefore, we assume an exogenous process for consumption growth and use the Euler equation:

$$E_t\left[\exp\left(m_{t+1}+r_{t+1}\right)\right]=1, \tag{3}$$

which holds for any continuous return $r_{t+1}=\log(R_{t+1})$, including the one on the wealth portfolio, to solve for the unobserved wealth-to-consumption ratio in the model.

3.2 Consumption and Inflation Dynamics

A novel ingredient of this model is that the consumption and inflation dynamics are subject to regime shifts. For notational brevity and expositional ease, we first specify the dynamics of consumption and inflation dynamics in a rather general VAR structure with regime shifts, and then provide the specific version of the dynamics that is our focus.

We assume there are S regimes that govern the dynamic properties of the n-dimensional state vector $Y_t \in R^n$. The regime variable s_t is a S-state Markov process, with the probability of switching from regime $s_t=j$ to $s_t=k$ given by a constant $\pi^{jk}, 0\leq j,k\leq(S-1)$,with $\sum_{k=0}^{S-1}\pi^{jk}=1$, for all j. Agents are presumed to know both the current and past histories of the state vector and the regime the economy is in. Thus, the expectation $E_t\left[\cdot\right]$ is conditioned on the information set I_t, generated by $\{Y_{t-l,}s_{t-l}:l\geq 0\}$. We use the notation $E^{(j)}\left[\cdot\right]$ to denote the unconditional mean of a random variable under the assumption of a single-regime economy governed by the parameters of regime j.

The Markov process governing regime changes is assumed to be conditionally independent of the Y process, such that $f\left(Y_{t+1}|Y_{t-l}:l\geq 0,s_t=j,s_{t+1}=k\right)=f\left(Y_{t+1}|Y_{t-l}:l\geq 0,s_t=j\right)$. In addition, given $s_t=j$, The state vector of the economy Y_{t+1} follows a VAR that is driven by both Gaussian and demeaned Gamma-distributed shocks:

$$Y_{t+1}=\mu(j)+F(j)Y_t+G_t(j)\varepsilon_{t+1}+\omega^j_{t+1}. \tag{4}$$

Here $\varepsilon_{t+1} \sim N(0, I)$ is the vector of Gaussian shocks, and ω^j_{t+1} is the vector of demeaned Gamma-distributed shocks. The detailed parameterization of the Gamma distribution will be provided later. To put the dynamics into an affine class, we impose an affine structure on G^j_t:

$$G_t(j)G'_t(j) = h(j) + \sum_i H_i(j)Y_{t,i}, \tag{5}$$

in which $h(j) \in R^{n\times n}$, $H_i(j) \in R^{n\times n}$, and i denotes the i-th element of state vector Y_t.

In the calibration section of this paper as well as later sections, we focus on a particular specification of (4). This specification is a generalized LRR model that incorporates regime shifts and non-neutrality of the expected inflation factor. Here we give an overview of this generalized LRR model and map it into the general framework in (4). Further details are also provided in the calibration section.

We specify the state dynamics as:

$$Y_{t+1} = \begin{pmatrix} \Delta c_{t+1} \\ \pi_{t+1} \\ x_{t+1} \\ z_{t+1} \\ \sigma^2_{t+1} \\ \Delta d_{t+1} \end{pmatrix}, \mu(j) = \begin{pmatrix} \mu^j_c \\ \mu^j_\pi \\ 0 \\ 0 \\ \left(\sigma^j_c\right)^2 (1-\nu_\sigma) \\ \mu^j_d \end{pmatrix}, F(j) = \begin{pmatrix} 0 & 0 & 1 & \tau_z & 0 & 0 \\ 0 & 0 & \tau_x & 1 & 0 & 0 \\ 0 & 0 & \nu_x & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_z & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu_\sigma & 0 \\ 0 & 0 & \phi & \phi\tau_z & 0 & 0 \end{pmatrix}, \tag{6}$$

in which the vector of Gaussian shocks $\varepsilon_{t+1} = (\varepsilon_{c,t+1}, \varepsilon_{\pi t+1}, \varepsilon_{x,t+1}, \varepsilon_{z,t+1}, 0, \varepsilon_{d,t+1}) \sim N(0, I)$ and $\omega^j_{t+1} = \left(0, 0, 0, 0, \omega^j_{\sigma,t+1}, 0\right)$. $\omega^j_{\sigma,t+1}$ follows a demeaned Gamma distribution: $\omega^j_{\sigma,t+1} = \widetilde{\omega}^j_{\sigma,t+1} - E\left(\widetilde{\omega}^j_{\sigma,t+1}\right)$. The Gamma distribution of $\widetilde{\omega}^j_{\sigma,t+1}$ is characterized by two parameters, so we specify the mean and volatility of the volatility shocks as:

$$E\left(\widetilde{\omega}_{\sigma,t+1}^{j}\right) = \left(\sigma_c^j\right)^2(1-\nu_\sigma), \quad (7)$$

$$Var\left(\widetilde{\omega}_{\sigma,t+1}^{j}\right) = \sigma_{c\omega}^2. \quad (8)$$

Therefore, we have $E\left(\omega_{\sigma,t+1}^{j}\right)=0$, and $Var\left(\omega_{\sigma,t+1}^{j}\right)=\left(\sigma_{c\omega}^{j}\right)^2$.

The first two components, Δc_{t+1} and π_{t+1}, denote the consumption growth and inflation. x_{t+1} and z_{t+1} are long-run expected growth and expected inflation factors. The term $(\mu_c + x_t + \tau_z z_t)$ is the conditional expectation of consumption growth in which x_t is a small but persistent expected consumption factor that captures long-run risks in consumption and dividend growth, as in the standard LRR model. Similarly, the term z_t is a small but persistent expected inflation factor that captures long-run risks in inflation. [1]

The parameter $\tau_z \neq 0$ leads to a non-neutral LRR model; in other words, the expected inflation factor feeds back to the real economy (i.e., the consumption process). For a typical parameter of $\tau_z < 0$, it means that a positive expected inflation factor will lower the future expected consumption growth. In this case, the long-run inflation risk is priced, and risk compensation for this risk factor is embodied in the risk premium, even for real assets.

The term Δd_{t+1} is logarithm dividend growth, which is defined as a leveraged process of Δc_{t+1}, with a leverage parameter $\phi > 1$. Thus, the dividend growth is more sensitive to x_t and z_t than consumption growth.

In order to guarantee that the variances always stay positive, we assume that the dynamics of volatility σ_{t+1}^2 follows an autoregressive Gamma process, following Barndorff-Nielsen and Shephard (2001) and Bansal and Shaliastovich (2013). The innovations in volatility process $\omega_{\sigma,t+1}^{j}$ follow a demeaned Gamma distribution. This specification will generate very similar asset pricing results

[1] Note that x_t and z_t are only part of the stochastic expected consumption and inflation, respectively, which is different with the standard LRR model. We call x_t and z_t expected growth and inflation factors, respectively, throughout this paper.

as a Gaussian volatility shock.

We set the conditional variance-covariance matrix of the Gaussian shocks to be $G_t(j)\,G_t(j)' = G_\sigma(j)\,G_\sigma(j)'\,\sigma_t^2 = H_\sigma(j)\,\sigma_t^2$, in which:

$$G_\sigma(j) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_\pi & 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_x & \rho_{xz}^j \varphi_z & 0 & 0 \\ 0 & 0 & 0 & \varphi_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \rho_{dc} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

The terms φ_π, φ_x, φ_z are regime-independent constants and denote the relative magnitudes of inflation volatility shock, long-run consumption shock, and long-run inflation shock with respect to short-run consumption shocks. This is a simplification assumption tying the dynamics of four Gaussian shocks to the same factor. It is straightforward to extend the model by allowing for multiple factors to derive different volatility shocks, although, we do not pursue this additional complication in this paper.

The parameter ρ_{xz}^j is regime-specific and captures the different correlations between long-run consumption and inflation shocks in different economic regimes. This specification closely follows the regime switching dynamic correlation ($RSDC$) model in Pelletier (2006). We can decompose the covariances into standard deviations and correlations. The standard deviations follow a continuous stochastic volatility process. In addition, the correlation follows a regime-switching model: it is constant within a regime, but different across regimes. This setup can be seen as a midpoint of the constant conditional correlation (CCC) model of Bollerslev (1990) and the dynamic conditional correlation (DCC) model of Engle (2002). An important advantage of the $RSDC$ specification is that it allows for tractability of the general equilibrium model within the affine structure world (after some log-linearization approximation), while also being able to generate time-varying correlations of risk factors.

4 Model Solutions and Intuitions

4.1 Within-regime Intuitions

Before going through the full solution of the full-blown LRR model with regime switching that we just discussed, we establish some within-regime intuitions. Specifically, we solve the model by assuming a single-regime economy governed by the parameters of regime j, for $j = 1, ...S$.

To generate an analytical solution for the model, we log-linearize the return on consumption asset to solve for the equilibrium discount factor and asset prices. In equilibrium, the wealth-to-consumption ratio, v_t, is linear in states:

$$v_t = A_0 + A_x x_t + A_z z_t + A_\sigma \sigma_t^2. \tag{10}$$

Using the Euler equation (3) and the assumed dynamics of consumption growth and inflation, we derive the solutions for coefficients A_x, A_z and A_σ :

$$A_x = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho_x}, \tag{11}$$

$$A_z = \frac{\left(1-\frac{1}{\psi}\right)\tau_z}{1-\kappa_1\rho_x}, \tag{12}$$

$$A_\sigma = \frac{(1-\gamma)\left(1-\frac{1}{\psi}\right)\left[1+\kappa_1^2\left\{\left(\frac{1}{1-\kappa_1\nu_x}+\frac{\tau_z\rho_{zx}}{1-\kappa_1\nu_z}\right)^2\varphi_x^2+\left(\frac{\rho_{xz}}{1-\kappa_1\nu_x}+\frac{1}{1-\kappa_1\nu_z}\right)^2\varphi_z^2\right\}\right]}{2(1-\kappa_1\nu_\sigma)}. \tag{13}$$

The details of the model solution and the expression for the endogenous log-linearization coefficients are provided in Appendix B.

It follows that A_x is positive if the IES, ψ, is greater than one. In this case, the intertemporal substitution effect dominates the wealth effects. In response to higher expected growth, agents buy more assets, and consequently the wealth-to-consumption ratio rises. In the standard power utility model with risk aversion larger than one, the IES is less than one, and hence A_x is negative; in turn, a rise in expected growth potentially lowers asset valuations. That is, the wealth effect dominates the substitution effect.

The coefficient A_z measures the sensitivity of wealth-to-consumption ratio to fluctuations of expected inflation factor. Consider the case in which IES, ψ, is greater than one: the sign of A_z is determined by τ_z, which captures the response of expected growth factor on the expected inflation. When typically $\tau_z < 0$, that is, a high expected inflation factor leads to a low expected consumption growth, the wealth-to-consumption ratio responds negatively ($A_z < 0$). In the LRR literature, both Bansal and Yaron (2004) and Bansal and Shaliastovich (2013), for example, set τ_z to zero. In these setup, the fluctuations of expected inflation does not feed back to the real economy (i.e., consumption growth), and thus does not affect real asset allocations and prices. In our setup ($\tau_z \neq 0$), the long-run expected inflation factor does affect the real economy, and we call it a non-neutral model.

The coefficient A_σ measures the sensitivity of the wealth-to-consumption ratio to volatility fluctuations. If the IES and risk aversion are larger than one, then the loadings are negative. In this case, a rise in consumption or expected growth volatility lowers asset valuations and increases the risk premium on all assets.

Using the equilibrium condition for the wealth-to-consumption ratio, we can provide an analytical expression for the pricing kernel:

$$\begin{aligned} m_{t+1} =& m_0 + m_x x_t + m_z z_t + m_\sigma \sigma_t^2 \\ & - \lambda_c \sigma_t \varepsilon_{c,t+1} - \lambda_x \varphi_x \sigma_t \varepsilon_{x,t+1} - \lambda_z \varphi_z \sigma_t \varepsilon_{z,t+1} - \lambda_\omega \omega_{\sigma,t+1} \end{aligned} \tag{14}$$

In particular, the conditional mean of the pricing kernel is affine in state variables x_t, z_t and σ_t^2, for which the loadings m_0, m_x, m_z and m_σ depend on model and preference parameters, as described in Appendix B.

The innovations in the pricing kernel are very important for thinking about risk premia. The magnitudes of the risk compensation depend on the market prices of short-run and long-run consumption and inflation risks, as well as volatility risks, which we denote as $\lambda_c, \lambda_x, \lambda_c$ and λ_ω respectively. The market prices of systematic risks can be expressed in terms of underlying preferences and parameters that govern the evolution of consumption growth and inflation:

$$\begin{aligned}\lambda_c =&\gamma, \\ \lambda_x =&(1-\theta)\kappa_1 A_x, \\ \lambda_z =&(1-\theta)\kappa_1 \left(A_x \rho_{xz} + A_z\right), \\ \lambda_\omega =&(1-\theta)\kappa_1 A_\sigma.\end{aligned} \tag{15}$$

The compensation for the short-run consumption risks is standard and given by the risk-aversion coefficient γ. In the special case of power utility, $\gamma = \frac{1}{\psi}$, the risk compensation parameters λ_x, λ_z, and λ_ω are zero, and the IMRS collapses to standard power utility specification:

$$m_{t+1}^{CRRA} = \log \delta - \gamma \Delta c_{t+1}. \tag{16}$$

With power utility there is no separate risk compensation for long-run growth, inflation risks, and volatility risks; meanwhile, with generalized preferences, these risks are priced. The pricing of long-run and volatility risks is an important feature of the long-run risks model.

When agents prefer early resolution of uncertainty, $\theta < 1$, (i.e. $\gamma > \frac{1}{\psi}$), the price of long-run consumption risks λ_x is positive, and the price of volatility risks λ_ω is negative. In other words, states with low expected growth or high volatility are bad states and discounted more heavily. It is important to note that the price of long-run inflation risks λ_z is intimately related to ρ_{xz}, which captures the sensitivity of expected consumption factor to the innovations of expected inflation. In particular, when $\rho_{xz} < 0$ (i.e., positive news in expected consumption factor predicts a decrease in long-run expected inflation), λ_z is negative; in contrast, when $\rho_{xz} > 0$, λ_z can switch its sign and become positive. In the model, we allow for this correlation parameter ρ_{xz} to switch signs, which is the key channel for getting sign-switching market prices of long-run inflation risk and, in turn, sign-switching nominal bond risk premium.

The discount factor used to price nominal payoff is given by:

$$m^{\$}_{t+1} = m_{t+1} - \pi_{t+1}. \tag{17}$$

The solution to the nominal discount factor is also affine in the state variables, and nominal market prices of risks depend on the real prices of risks and inflation sensitivity to both short- and long-run consumption and inflation news.

Given the solutions to the real and nominal discount factor, we observe that the yields on real and nominal bonds satisfy:

$$y_{t,n} = \frac{1}{n}\left(B_{0,n} + B_{x,n}x_t + B_{z,n}z_t + B_{\sigma,n}\sigma_t^2\right), \tag{18}$$

$$y^{\$}_{t,n} = \frac{1}{n}\left(B^{\$}_{0,n} + B^{\$}_{x,n}x_t + B^{\$}_{z,n}z_t + B^{\$}_{\sigma,n}\sigma_t^2\right). \tag{19}$$

The bond coefficients, which measure the sensitivity of bond prices to the aggregate risks in the economy, are pinned down by the preference and model parameters. As shown in Appendix B, the real yields respond positively to the expected growth factor, $B_{x,n} > 0$, and respond negatively to the volatility state, $B_{\sigma,n} < 0$. When a high expected inflation factor indicates lower future consumption ($\tau_z < 0$), the real yields load negatively on expected inflation factor, $B_{z,n} < 0$.

The one-period expected excess return on a real bond with 2 months to maturity can be written in the following form:

$$
\begin{aligned}
rp_{t+1}^{(2)} =& E_t rx_{t+1}^{(2)} + \frac{1}{2} Var_t rx_{t+1}^{(2)} \\
=& -\frac{1}{\psi}\left(\gamma - \frac{1}{\psi}\right)\kappa_1 \left(\begin{array}{c} \frac{\varphi_x^2 + \rho_{xz}^2 \varphi_z^2}{1-\kappa_1 \nu_x} + \frac{\tau_z^2 \varphi_z^2}{1-\kappa_1 \nu_z} \\ +\tau_z \rho_{xz} \left(\frac{1}{1-\kappa_1 \nu_x} + \frac{1}{1-\kappa_1 \nu_z}\right) \varphi_z^2 \end{array} \right) \sigma_t^2 - B_{\sigma,1} \lambda_\sigma \left(\sigma_{c\omega}\right)^2
\end{aligned} \tag{20}
$$

The last term, $-B_{\sigma,1}\lambda_\sigma (\sigma_{c\omega})^2$, is the component of the real bond premium attributable to volatility risks. As shown in Appendix B, $-B_{\sigma,1}$ is the beta of the real bond return with respect to volatility innovations, which is positive. Since the market price of volatility risks λ_σ is negative, the volatility risks always contribute negatively to the real bond premium. If the correlation of expected consumption and expected inflation shocks is zero, $\rho_{xz} = 0$, the real bond risk premium from long-run consumption and inflation risks is always negative. When the expected consumption factor decreases at the time of high expected inflation factor (i.e., $\tau_z < 0$), then negative correlations of expected consumption and inflation shocks (i.e., $\rho_{xz} < 0$) will decrease the real bond premium even further. On the other hand, a positive correlation of these shocks can increase the bond risk premium. We identify similar implications for real bond premium at the long end as well.

For nominal bonds, as long as the negative effect of the inflation factor on long-run consumption is not very strong (τ_z very negative), which is the case in our calibration, we find that the nominal yields load positively on the inflation factor, $B_{z,n}^{\$}$. Intuitively, with high expected inflation, investors require higher yields for nominal bonds to compensate for the erosion of purchasing power. That said, nominal bond returns load negatively on inflation shocks (negative beta with respect to inflation risk).

We can express nominal bond risk premium as real bond risk premium plus an additional component capturing the risk compensation for inflation shocks:

$$rp_{t+1}^{\$(2)} - rp_{t+1}^{(2)} = -\left(\gamma - \frac{1}{\psi}\right)\kappa_1 \begin{pmatrix} \frac{\rho_{xz}\varphi_z^2}{1-\kappa_1\nu_x} + \frac{\tau_z\varphi_z^2}{1-\kappa_1\nu_z} \\ +\tau_x\left(\frac{\varphi_x^2+\rho_{xz}^2\varphi_z^2}{1-\kappa_1\nu_x} + \frac{\tau_x\rho_{xz}}{1-\kappa_1\nu_z}\right)\varphi_z^2 \end{pmatrix}\sigma_t^2 \tag{21}$$

The correlation parameter ρ_{xz} is a very important determinant of the risk premium for nominal bonds. Its regime-dependent feature is the driving force that leads to sign-switching bond risk premium. Under the typical parameterization of $\tau_z < 0$ and $\tau_x \leq 0$, when $\rho_{xz} < 0$, the inflation risk premium is positive; however, when $\rho_{xz} > 0$, the inflation risk premium turns negative.

The sign-switching correlation ρ_{xz} also leads to sign-switching stock and bond correlations. As we discuss earlier, nominal bond returns always have negative beta with respect to inflation shocks. In contrast, because stock is a levered consumption claim, it loads negatively on inflation shocks when $\rho_{xz} < 0$ and loads positively on inflation shocks when $\rho_{xz} > 0$. This generates a positive correlation between stock and bond returns when $\rho_{xz} < 0$ and a negative correlation between stock and bond returns when $\rho_{xz} > 0$.

4.2 Characterizations of Different Regimes

A salient feature of the model is that we allow for regime shifts in exogenous consumption and inflation dynamics. In particular, with respect to calibration, we allow for 3 regimes – expansion, contraction and deep recession regimes – and summarize the regime-specific elements in Table 1. We first discuss the implications of different regime-specific elements, and then give economic interpretations of the three regimes.

First, we set the nominal-real correlation, ρ_{xz}, to switch signs, such that $\rho_{xz} < 0$ in expansion and contraction regimes, while $\rho_{xz} > 0$ in deep recession regimes. According to (15), the market price of long-run inflation risks can switch signs, such that the price is negative with negative correlation, and vice versa. On the other hand, the beta of nominal bond returns with respect to long-run inflation risk innovations stays negative. As a whole, long-run inflation risks contribute positively to the nominal bond risk premium in expansion and contraction regimes, but contribute negatively in deep-recession regimes. The time-varying correlation also causes the equity beta of long-run inflation risks to switch sign, which generates sign-switching correlations between stock and nominal bond returns. Specifically, in deep recessions, the stock and bond correlation is negative, but remains positive in other times. This is consistent with the empirical evidence we highlighted in the introduction of this paper. These intuitions are summarized in Table 2. In sum, this time-varying nominal-real correlation structure is the key channel for generating sign-switching nominal bond risk premium as well as generating stock-bond correlations.

Second, we allow for different levels of consumption growth and inflation. In equilibrium, higher μ_c, which is the unconditional mean of consumption growth, implies higher real and nominal bond yields at all maturities simultaneously, while μ_π, which is the unconditional mean of inflation, affects nominal yields only. Higher μ_π indicates higher nominal yields at all maturities. In sum, the consumption growth and inflation levels, μ_c and μ_π, are "level" factors to term structures of interest rates. The former shifts both real and nominal yield curves, while the latter only affects the nominal side. This channel is very important for generating the shifts of yield curves in different economic regimes.

Third, macroeconomic uncertainty, as captured by consumption and inflation volatility (tied to each other in the parsimonious specification of this paper), are allowed to be time-varying. This time-varying feature of uncertainty present a two-fold observation. First, within each regime, the volatility follows an autoregressive Gamma process. This is important for producing enough variations in equity and bond risk premium and for helping to replicate the violations of the expected hypothesis of bond returns. The intuition is similar to Bansal and Shaliastovich (2013). Second, we allow for regime-specific uncertainty levels. We specify a low mean level in expansion, but specify a higher mean level in bad states. This channel corresponds to a counter-cyclical property of stock volatility, and is very important for generating higher equity premium in contraction/deep-recession regimes, which is consistent with the empirical findings of Lustig and Verdelhan (2012). Furthermore, the mean level of volatility also constitutes a "slope" factor of the yield curve, as the real (nominal) yield curve slope is determined by the risk premium of the long-term real (nominal) bond, which is proportional to volatility levels, as we show in Appendix B. Different levels of macroeconomic volatility will also alter the levels of real and nominal bond yields, due to households' precautionary saving motives.

We broadly classify the economy into three regimes: expansion, contraction, and deep-recession. In the expansion regime, we have high consumption growth, a medium level of inflation, and low uncertainty (which is measured by consumption and inflation volatility). In the contraction regime, we have lower growth rate and higher uncertainty; the inflation level is also higher in this regime. One can think of this regime as a stagflation regime, in which low growth and high inflation coexist. A typical sample episode is that of the late 1970s and early 1980's. In the deep-recession regime, we have the lowest growth and highest level of uncertainty. As opposed to regular contraction, this regime has very low inflation, since deflation rather than inflation is more of a concern at this time. Another key ingredient that is different across these three regimes is the nominal-real correlation, which refers to the correlation between shocks to expected growth and inflation factor in the model. In the first two regimes, positive news to expected inflation factor indicates lower future expected growth; however, in the deep recession regime, the relationship is just the opposite. This ingredient is key for generating large movements (and potentially sign-switching) in nominal bond risk premium as well as stock-bond correlations.

4.3 Solutions to Long-run Risks Model with Regime Shifts

We now solve for the equilibrium price process of the model with regime switching. The mechanism in the full model that generates sign-switching stock and bond returns is similar to that we discussed in Section 4.1 and Section 4.2, except that adding regime shifts introduces regime-switching risk premium. In this section, we only lay down the basic solution method and leave the detailed discussion of regime-switching risk and its pricing to the quantitative Section 5.2.1.

To price assets, we must first solve for the return on a consumption claim, $r_{c,t+1}$, as it appears in the pricing kernel itself. We denote the logarithm of the wealth-to-consumption ratio at a given time t and state $s_t = j$ by $v_{c,t}(j)$, and we use the Campbell and Shiller (1988) log-linearization to linearize $r_{c,t+1}(j,k)$, which depends on two consecutive states $s_t = j$ and $s_{t+1} = k$, around the unconditional means of $v_t(j)$ and $v_{t+1}(k)$, respectively:

$$r_{c,t+1}(j,k) = \kappa_{c,0}(k) + \kappa_{c,1}(k)\, v_{c,t+1}(k) - v_{c,t}(j) + \Delta c_{t+1}(j). \tag{22}$$

A similar approach is taken by Bansal and Yaron (2004), and Bansal, Kiku, and Yaron (2007a) for a standard LRR model without regime switching. We then conjecture that given the current state $s_t = j$, the log wealth-to-consumption ratio is affine in the state vector:

$$v_t(j) = A_0(j) + A' Y_t(j), \tag{23}$$

in which $A(j) = (A_c(j), A_\pi(j), A_x(j), A_z(j), A_\sigma(j), A_d(j))'$ is a vector of pricing coefficients, which are regime-specific. Substituting (23) into (22) and then substituting (22) into the Euler equation (3) gives the equation in terms of A, A_0 and the state variables. The expectation on the left-hand side of the Euler equation can be evaluated analytically, as shown in Appendix C. Since any predictive information in Δc_t, π_t and Δd_t is contained in x_t and z_t, they have no effects on $v_t(j)$. Therefore, $A_c(j) = A_\pi(j) = A_d(j) = 0$.

Having solved for $r_{c,t+1}(j,k)$, we substitute it into m_{t+1}, which also depends on two consecutive states $s_t = j$ and $s_{t+1} = k$, to obtain an expression for the log pricing kernel at time $t+1$:

$$m_{t+1}(j,k) = m_0(j,k) + m_1(j,k)' Y_t - \Lambda(k)' \left(G_t(j)\, \varepsilon_{t+1} + \omega_{t+1}^{j} \right). \tag{24}$$

The loadings $m_0(j,k)$, $m_1(j,k)$ and $\Lambda(k)$ are provided in Appendix C.

We then solve for the market return. A share in the market is modeled as a claim on a dividend stream with growth process given by Δd_{t+1}. To solve for the price of a market, we proceed along the same line as that for the consumption claim and solve for the price-to-dividend ratio of the market at time t in regime j, denoted as $v_{m,t}^{j}$, using Euler equation (3). To do this, we log-linearize the returns on the market, $r_{m,t+1}$, which depends on two consecutive states, $s_t = j$ and $s_{t+1} = k$, around the unconditional means of $v_{m,t}(j)$ and $v_{m,t+1}(k)$, respectively:

$$r_{m,t+1}(j,k) = \kappa_{0,m}(k) + \kappa_{1,m}(k)\, v_{m,t+1}(k) - v_{m,t}(j) + \Delta d_{t+1}(j). \tag{25}$$

We conjecture that $v_{m,t}(j)$ is affine in the state variables:

$$v_{m,t}(j) = A_{0,m}(j) + A_{1,m}(j)' Y_t, \tag{26}$$

in which $A_{1,m}(j) = (A_{c,m}(j), A_{\pi,m}(j), A_{x,m}(j), A_{z,m}(j), A_{\sigma,m}(j), A_{d,m}(j))'$ is the vector of pricing coefficients that are regime-dependent. For similar reasons as in wealth-to-consumption ratios, we obtain $A_{c,m}(j) = A_{\pi,m}(j) = A_{d,m}(j) = 0$. Substituting the expression for $v_{m,t}(j)$ into the linearized return, we obtain an expression for $r_{m,t+1}(j,k)$ in terms of Y_t and its innovations:

$$r_{m,t+1}(j,k) = J_{0,m}(j,k) + J_{1,m}(j,k)' Y_t + \beta_d(k)' \left(G_t(j)\, \varepsilon_{t+1} + \omega_{t+1}^{j} \right), \tag{27}$$

for which the loadings of $J_0(j,k)$, $J_1(j,k)$ and $\beta_d(k)$ are provided in Appendix C.

The equilibrium real and nominal yield are affine in the state variables. Indeed, in Appendix C, we show that real and nominal yields at time t given the state $s_t = j$ satisfy:

$$y_{t,n}(j) = \frac{1}{n}\left[B_{0,n}(j) + B_n(j)' Y_t \right], \tag{28}$$

$$y^{\$}_{t,n}(j) = \frac{1}{n}\left[B^{\$}_{0,n}(j) + B^{\$}_{n}(j)' Y_t\right]. \tag{29}$$

The bond coefficients, which measure the sensitivity of bond prices to the aggregate risks in the economy, are pinned down by the preference and model parameters. The expressions of the loadings are presented in Appendix C.

5 Calibration and Quantitative Results

5.1 Calibration Parameters

We calibrate the monthly subjective discount factor δ to be 0.9989. The risk-aversion coefficient is set at $\gamma = 10$. As in Bansal and Yaron (2004), we focus on an IES of 1.5. As we discussed earlier, an IES value larger than one is important for the quantitative results. Bansal, Kiku, and Yaron (2007b) document that the asset valuations fall when consumption volatility is high, which is consistent only with $\psi > 1$.

We follow the standard LRR literature to calibrate the parameters for consumption and inflation outlined in equation (4) at a monthly frequency and time-aggregate the output from monthly simulations to match the key aspects of annual consumption growth and inflation rates in U.S. from 1930 to 2021. We specify a high consumption growth, median inflation, and low volatility for expansions; a median consumption growth, high inflation, and median volatility for contractions; and a low consumption growth, low inflation, and high volatility for deep recessions. We summarize all the parameter values in Table 3. We report the model-implied consumption and inflation moments, in Table 6, which is based on a very long simulation of monthly data aggregated to annual horizons. As we show in Table 6, the model can match very well the salient features of the consumption and inflation data.

To calibrate regime-switching probabilities, we use $j = 0, 1, 2$ to denote deep recession, contraction and expansion regimes, respectively. We assume $\pi^{01} = 0$ and $\pi^{10} = 0$. Therefore, two-types of recessions cannot switch from each other. We use the length of NBER dated business cycles and recession frequencies to calibrate the transition probability matrix. Our calibration is based on 18 recessions from $1919 - 2021$, in which we consider two events as deep recessions: (1) August 1929 to March 1933 and (2) December 2007 to June 2009; we count all other 16 events as regular recessions[2]. Tables 4 and 5 shows the calibration of the transition probability matrix and its implications for unconditional probability and average duration of each regime. The model-implied moments match the data counterparts quite well. We can see that because the unconditional probability of deep recessions is less than that of contractions, conditioning on being in an expansion, the probability of switching to a deep recession is less than switching to a contraction (0.21% vs. 1.63%). Because the average duration of deep recessions is longer, conditioning on being in a deep recession, the probability of remaining in the deep recession is greater than that of contractions (96.72% vs. 90.70%).

5.2 Quantitative Results

5.2.1 Unconditional Asset Pricing Moments

The model-implied unconditional asset pricing moments match the data quite well. Table 7 reports the model performance in terms of unconditional moments on the bond and stock markets. As we can see, the model, on average, generates a downward-sloping real yield curve, but generates a upward-sloping nominal yield curve. This is because long-term real bonds act as a hedge to long-run consumption shocks, and investors require a lower expected return holding long-term real bonds. In contrast, because long-term nominal bonds give lower real returns in face of a positive inflation shock and inflation risk is negatively priced most of the time, investors require higher

[2]The recent COVID-19 recession does cause significant drops in GDP, but it features both low growth and high inflation because of the shortage of supplies due to social distancing. Therefore, we classify it as a regular recession in our calibration.

expected returns holding long-term nominal bonds. The model generates an average nominal yields around 4.8% and a 5-year minus 1-year term premium of 0.56%, which are close to the data. On the equity market, the model can also generate a high equity premium of 8.34% as is the case in the data. The success of our model in quantitatively reproducing a high risk premium lies in two ingredients: (1) risk premium from the persist long-run component, and (2) risk premium from regime shifts.

To show the pricing of regime shifts in our model, we calculate the prices and risk premia of regime contingent claims. A k regime contingent claim is such an asset that pays off 1 if the regime in the next period is k and pays off 0 otherwise. We denote $p_{j,k}$ as the price of k regime contingent claim given the current regime is j, and denote $rp_{j,k}$ as the risk premium of the asset. For instance, $p_{2,0}$ and $rp_{2,0}$ are the price and risk premium of an asset that pays off 1 if the regime in the next period is a deep recession, given that today's regime is expansion. We base our calculation on the unconditional mean of state variables in each regime and report our results in Table 8.

As shown in Table 8 Panel A, the price of a contingent claim on expansion is usually much smaller than its physical transition probability. For instance, the transition probability from deep recession to expansion is 3.28% while $p_{0,2} = 0.0167$. In contrast, the prices of contingent claims on deep recession and contraction are usually much larger than their physical transition probability. This is because claims on expansion are risky as they only pays off in good states while claims on deep recession and contraction are hedges to bad states. Panel B reports the annualized risk premium for each contingent claim. We observe that risks of regime shifts are associated with very high risk premium. For example, the claim on expansion condition on deep recession, $rp_{0,2}$, is associated with a risk premium of 1160.25%, and the claim on deep recession condition on expansion, $rp_{2,0}$, is associated with a negative premium of -664.12%. This is because the transition of regimes is a small probability event but its effect is huge and persistent. This generates large differences of marginal utility across different regimes, and in turn leads to a high risk premium. In this economy, risky assets that pay off well in expansion and badly in recessions, such as stocks, will be compensated with risk premium derived from regime-switching risk.

5.2.2 Dynamics of Asset Prices

Our model also generates realistic time-variations of stock and bond returns. Table 9 reports the model-implied yields conditional on different regimes. Consistent with Bansal and Zhou (2002), we see significant level shifts across different regimes. In the deep recession regime, the nominal yield is the lowest, mainly because both consumption growth and inflation levels are low, while the macro uncertainty is high. The nominal yield curve in the contraction regime is the highest, mainly because the inflation level is the highest at this regime.

In Table 10, we report the model-implied risk premia for nominal (real) bond returns as well as equity returns, conditional on different regimes. Following our earlier intuition, in the deep recession regime, the nominal bond premium is negative, while it is positive in other regimes, due to the sign-switching long-run inflation risk premium. We also get conditional stock-bond correlations that are consistent with those in Figure 1. Another feature worth mentioning is that conditional on the contraction regime, we have higher premia in both bond and equity compared those in expansion regime. This is consistent with the findings of Lustig and Verdelhan (2012).

In Table 11, we report the model implications for bond return predictability of Cochrane and Piazzesi (2005). Following their approach, we regress the average of the 1-year ahead bond excess returns (with maturity from 2 to 5 years) on the 1-year, 3-year and 5-year forward rates:

$$\frac{1}{4}\sum_{n=2}^{5} rx_{t+12,12n} = \gamma_0 + \gamma_1 f_{t,12} + \gamma_2 f_{t,36} + \gamma_3 f_{t,60} + error.$$

We extract a single bond factor $\widehat{rx}_{t,m} = \widehat{\gamma}_0 + \widehat{\gamma}_1 f_{t,12} + \widehat{\gamma}_2 f_{t,36} + \widehat{\gamma}_3 f_{t,60}$ from this regression, which is subsequently used to forecast bond excess returns at maturity from 2 to 5 years:

$$rx_{t+m,n} = const + b_{m,n}\widehat{rx}_{t,m} + error.$$

We replicate their work and extend the sample period to December 2021. The estimates $b_{m,n}$ are positive and increasing with horizons, and a single factor projection captures $12 - 15\%$ of the variation in bond returns. From the model, the slope coefficients in the second-stage regressions increase from 0.45 at 2-year maturity to 1.51 at 5 years, which matches very well with the data estimates. The model-implied R^2 in the projections is about 13%, which is also close to the data. This implies that the model does generate considerable time-variation of bond risk premium due to both time-varying volatility and regime-switching channels. The single-factor projections for real bonds delivers a very similar pattern for the second-stage coefficients. The R^2 in real regressions are quite substantial, although they decrease somewhat relative to their nominal counterparts. The predictability in our model comes from two sources. First, because volatility follows an autoregressive Gamma process, the risk premium fluctuates even within one regime. Second, as we can see from Table 10, the risk premium of nominal bonds changes significantly and even switches signs across different regimes. These two ingredients contribute to the high return predictability inside our model.

6 Conclusion

This paper studies the joint determinants of stock and bond returns in a Bansal and Yaron (2004) type of long-run risks framework. A novel ingredient of our model is allowing for regime shifts in consumption and inflation dynamics. In particular, the means, volatilities, and the correlation structure of consumption and inflation dynamics are regime-dependent. This rational expectations general equilibrium framework can (1) jointly match the dynamics of consumption, inflation and cash flow; (2) generate time-varying and sign-switching stock and bond correlations, as well as generating sign-switching bond risk premium; and (3) coherently explain another long list of salient empirical features in stock and bond markets, including time-varying equity and bond return premia, regime shifts in real and nominal yield curves, the violation of the expectations hypothesis of bond returns. The model also conveys the insight that the term structure of interest rates and stock-bond correlations are intimately related to business cycles, while long-run risks and volatility risks play a more important role in accounting for high equity premium than do business cycle risks.

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Tables and Figures

Table 1: Characterizations of Different Regimes

This table summarizes the characteristics of different regimes in the model.

	Expansion	Contraction	Deep-Recession
Nominal-real correlation, ρ_{xz}	Negative	Negative	Positive
Consumption level, μ_c	High	Medium	Low
Inflation level, μ_π	Medium	High	Low
Uncertainty, σ_c^2	Low	Medium	High

Table 2: Market Price of Risks, and Betas to Risk Innovations

This table summarizes the price of risks and betas in different regimes in the model.

Shocks	SR cons.	LR cons.	**LR Infl.**	Vol.	Overall Premium
Expansion regime					
Risk Price	+	+	**-**	-	
Beta-equity return	+	+	**-**	-	+
Beta-nominal-bond return	0	+	**-**	-	+
Deep Recession Regime					
Risk Price	+	+	**+**	-	
Beta-equity return	+	+	**+**	-	+
Beta-nominal-bond return	0	+/-	**-**	-	-

Table 3: Calibration Parameters

This table reports calibrated parameter values for the baseline model. The model is calibrated at monthly frequency; hence we report monthly parameter values. $j = 0, 1, 2$ denote deep-recession, contraction and expansion regimes, respectively.

Parameters		Parameter Value		
Preference Parameters				
Subjective discount factor	δ		0.9989	
IES	ψ		1.5	
Risk Aversion	γ		10	
Consumption and Inflation Dynamics: Regime-independent Parameters				
Persistence of x_t	ν_x		0.987	
Persistence of z_t	ν_z		0.987	
Persistence of σ_t^2	ν_σ		0.982	
$\sigma_{\pi,t}/\sigma_t$ ratio	φ_π		2	
$\sigma_{x,t}/\sigma_t$ ratio	φ_x		0.03	
$\sigma_{z,t}/\sigma_t$ ratio	φ_z		0.05	
The loadings of Δc_{t+1} on z_t	τ_z		-0.2	
The loadings of π_{t+1} on x_t	τ_x		-1	
Volatility of Volatility	$\sigma_{c\omega}$		$2.66e^{-6}$	
Consumption and Inflation Dynamics: Regime-dependent Parameters				
Regimes	j	0	1	2
Consumption mean	μ_c	0.004	0.008	0.017
Inflation mean	μ_π	0.008	0.038	0.027
Correlations between x_t and z_t shocks	ρ_{xz}	0.5	-0.8	-0.5
Consumption volatility	σ_c	0.0075	0.0064	0.0040

Table 4: Transition Probability Matrix Calibration

This table reports the calibrated value of the transition probability matrix. The notation $j = 0, 1, 2$ denote deep-recession, contraction and expansion regimes, respectively.

	Deep-Recession ($j\prime = 0$)	Contraction ($j\prime = 1$)	Expansion ($j\prime = 2$)
$j = 0$	0.9672	0	0.0328
$j = 1$	0	0.9070	0.0930
$j = 2$	0.0021	0.0163	0.9816

Table 5: Model-Implied Regime Switching Features

This table reports the recession frequency and duration implied by the model. The data counterparts are based on the length of 18 NBER dated recessions from $1919 - 2021$. Two events (1) 1929/08 - 1933/03 and (2) 2007/12 - 2009/6 are considered deep-recessions. $j = 0, 1, 2$ denote deep-recession, contraction and expansion regimes, respectively.

	Unconditional Probability			Durations, months		
	$j=0$	$j=1$	$j=2$	$j=0$	$j=1$	$j=2$
Data	0.0503	0.1418	0.8079	30.5	10.75	54.44
Model	0.0509	0.1415	0.8076	30.5	10.75	54.44

Table 6: Model-Implied Unconditional Moments

This table reports the aggregate moments implied by the model. The data moments are based on annualized consumption and inflation data from $1930 - 2021$. Consumption includes non-durable expenditure and service from BEA. Inflation is deseasonalized CPI from the FRED datasets. The model-implied moments are computed from a long simulation of monthly model, and time aggregated to annual frequency. All the statistics reported in this table are annualized.

Variable	Data	Model
$E(\Delta c)$	1.95	1.96
$std(\Delta c)$	2.29	2.10
$AC1(\Delta c)$	0.34	0.61
$AC2(\Delta c)$	0.13	0.42
$E(\pi)$	3.12	3.08
$std(\pi)$	3.97	3.93
$AC1(\pi)$	0.61	0.56
$AC2(\pi)$	0.29	0.32

Table 7: Bond and Equity Markets

This table reports bond yields and equity premium implied by the model as well as the data counterparts. The equity returns are the CRSP value-weighted portfolio that comprises the stocks traded in the NYSE, AMEX and NASDAQ, from $1926-2021$. The nominal yields data are from the Fama-Bliss monthly datasets from June 1952 – Dec 2021. The model-implied moments are computed from a long simulation of monthly model, and time aggregated to annual frequency. All statistics reported in this table are annualized.

	1y	2y	3y	4y	5y
	Nominal Term Structure				
Mean - Data	4.56	4.75	4.93	5.08	5.19
Mean - Model	4.55	4.69	4.83	4.97	5.11
	Equity Returns				
	Data	Model			
Equity Premium	8.28	8.34			
Std Dev.	18.75	10.74			

Table 8: Prices and Risk Premium of Regime Contingent Claims

This table reports the prices and risk premium of regime contingent claims based on the unconditional mean of state variables in each regime. Risk premium is annualized, which is equal to monthly risk premium $\times 12$. The notation $j = 0, 1, 2$ denotes deep-recession, contraction and expansion regimes, respectively.

	Deep-Recession ($k = 0$)	Contraction ($k = 1$)	Expansion ($k = 2$)
	Prices: $p_{j,k}$		
$j = 0$	0.9835	$n.a.$	0.0167
$j = 1$	$n.a.$	0.9274	0.0726
$j = 2$	0.0046	0.02	0.9740
	Risk Premium: $rp_{j,k}$		
$j = 0$	-19.67%	$n.a.$	1160.25%
$j = 1$	$n.a.$	-26.39%	336.96%
$j = 2$	-664.12%	-222.06%	7.68%

Table 9: Model Implications – Bond Yield

This table reports the model implications of bond yields. The yields are reported by averaging the monthly yields conditional on the regime state variables. The yields are annualized.

	1y	2y	3y	4y	5y
Conditional on expansion regime					
Real Yield	1.30	0.99	0.69	0.41	0.14
Nominal Yield	4.64	4.78	4.92	5.06	5.19
Conditional on deep-recession regime					
Real Yield	−0.05	−0.27	−0.49	−0.71	−0.93
Nominal Yield	1.75	1.85	1.98	2.13	2.29
Conditional on contraction regime					
Real Yield	0.62	0.43	0.20	−0.03	−0.27
Nominal Yield	4.78	4.94	5.11	5.28	5.43

Table 10: Model Implications – Equity and Bond Premium

This table reports the model implications of equity and bond premium. The risk premia are reported by averaging 1-month nominal/real bond holding period excess returns and 1-month equity excess returns, conditional on regime state variables. The statistics are annualized.

	1y	2y	3y	4y	5y
Conditional on expansion regime					
Real-bond Holding Period Return Premium	−0.19	−0.74	−1.27	−1.77	−2.25
Nom-bond Holding Period Return Premium	0.08	0.35	0.65	0.95	1.25
Equity Premium	7.42		$corr(r_b^{(5)}, r_d)$		0.54
Conditional on deep-recession regime					
Real-bond Holding Period Return Premium	−0.33	−0.84	−1.31	−1.75	−2.14
Nom-bond Holding Period Return Premium	−1.18	−2.31	−3.34	−4.27	−5.10
Equity Premium	10.70		$corr(r_b^{(5)}, r_d)$		−0.42
Conditional on contraction regime					
Real-bond Holding Period Return Premium	−0.72	−1.42	−2.00	−2.49	−2.91
Nom-bond Holding Period Return Premium	0.88	1.81	2.64	3.36	4.00
Equity Premium	8.78		$corr(r_b^{(5)}, r_d)$		0.82

Table 11: Model Implications, Single Factor Projection

This table reports the nominal return predictability test in the U.S. bond market. Monthly observations of 1-5 year yields from June 1952 – Dec 2021 are from the Fama-Bliss datasets. We report the slope coefficients $b_{m,n}$ and R^2 in single latent factor regression $rx^{\$}_{t+m,n} = const + b_{m,n}\widehat{rx^{\$}}_{t,m} + error$, in which $rx^{\$}_{t+m,n}$ is an m-months excess return on a n-period bond, and $\widehat{rx^{\$}}_{t,m}$ corresponds to a single bond factor obtained from a first-stage projection of average bond returns on three forward rates. Model-implied slope coefficients and R^2 in single latent factor regressions are based on a very long simulation of 500 years' monthly observations. Standard errors are Newey-West adjusted with 10 lags.

n	2y	3y	4y	5y
	US data (1952 – 2021)			
Coeff.	0.43	0.86	1.23	1.47
Std. Dev.	0.11	0.22	0.31	0.38
R^2	0.12	0.14	0.15	0.14
	Model - Normal Regression			
Coeff.	0.45	0.85	1.20	1.51
Std. Dev.	0.05	0.10	0.15	0.19
R^2	0.127	0.130	0.130	0.129
	Model - Real Regression			
Coeff.	0.46	0.86	1.20	1.49
Std. Dev.	0.07	0.13	0.19	0.24
R^2	0.062	0.059	0.057	0.054

Fig. 1: Correlation between Stock and Nominal Bond Return

This figure plots the time-varying correlation between stock and long-term nominal bond returns, which is calculated based on a 3-year centered moving window of real monthly stock and bond returns. The sample covers Jan 1926 - Dec 2021. Shaded areas denote NBER recessions.

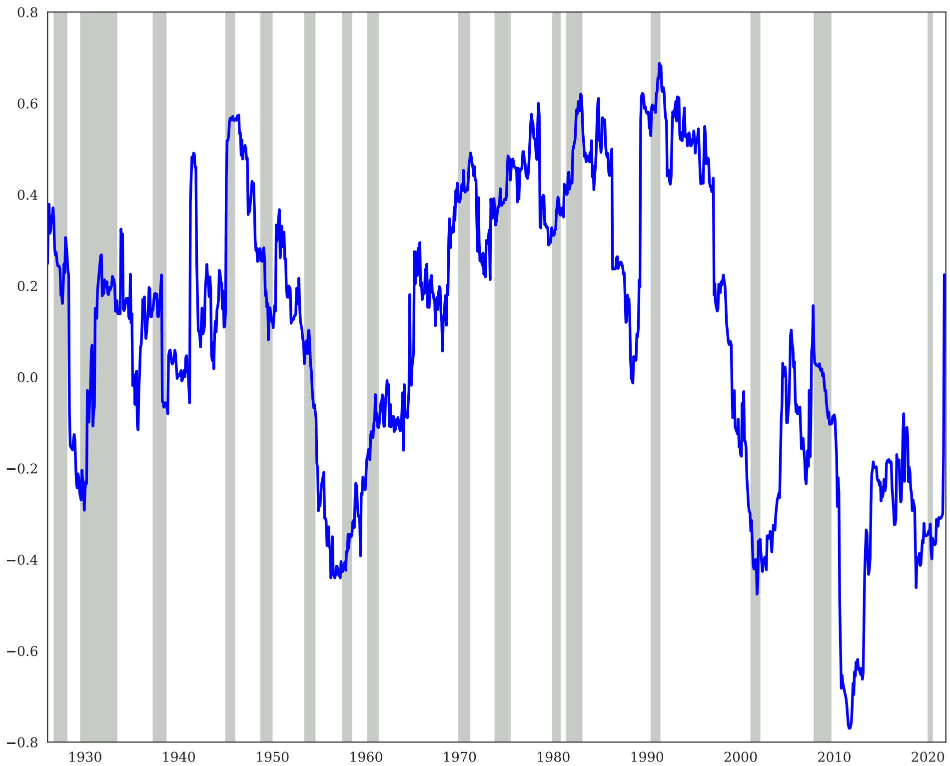


Fig. 2: CAPM Beta of Inflation

This figure summarizes the CAPM beta of inflation. We use a 5-year centered moving window of quarterly data and a first-order VAR for inflation, stock returns (real), and three-month treasury bill returns to calculate inflation shocks. Beta is defined as $\widehat{Cov_t}\left(\pi_t, r_{m,t}\right)/\widehat{Var_t}\left(r_{m,t}\right)$, in which π_t, $r_{m,t}$ are shocks to inflation and market returns respectively. The sample covers 1926Q1 to 2021Q4. Shaded areas denote NBER recessions.

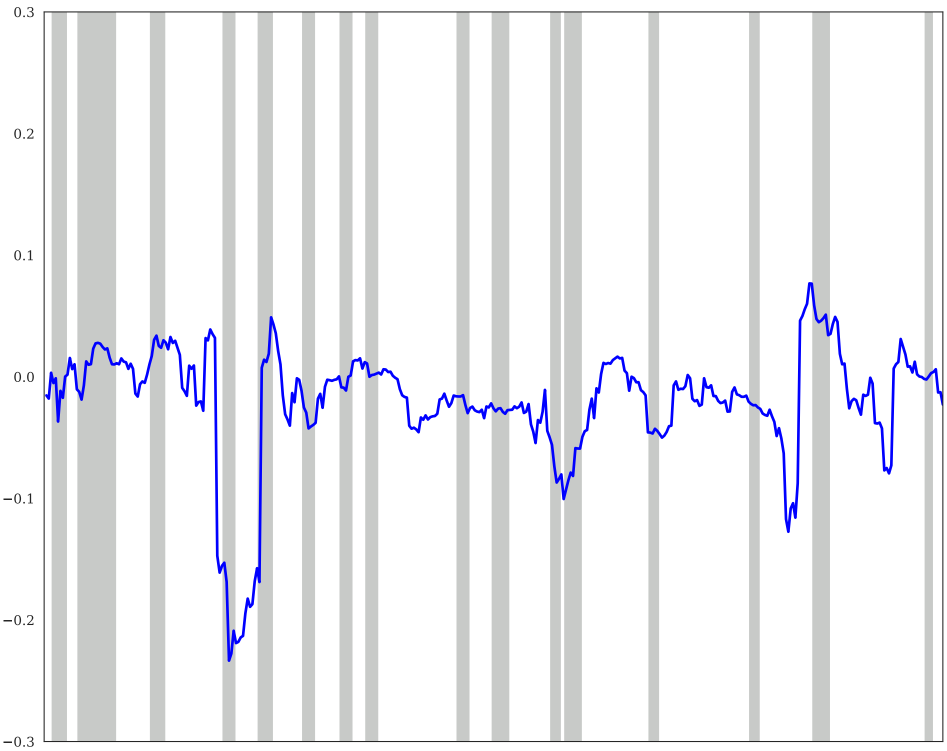
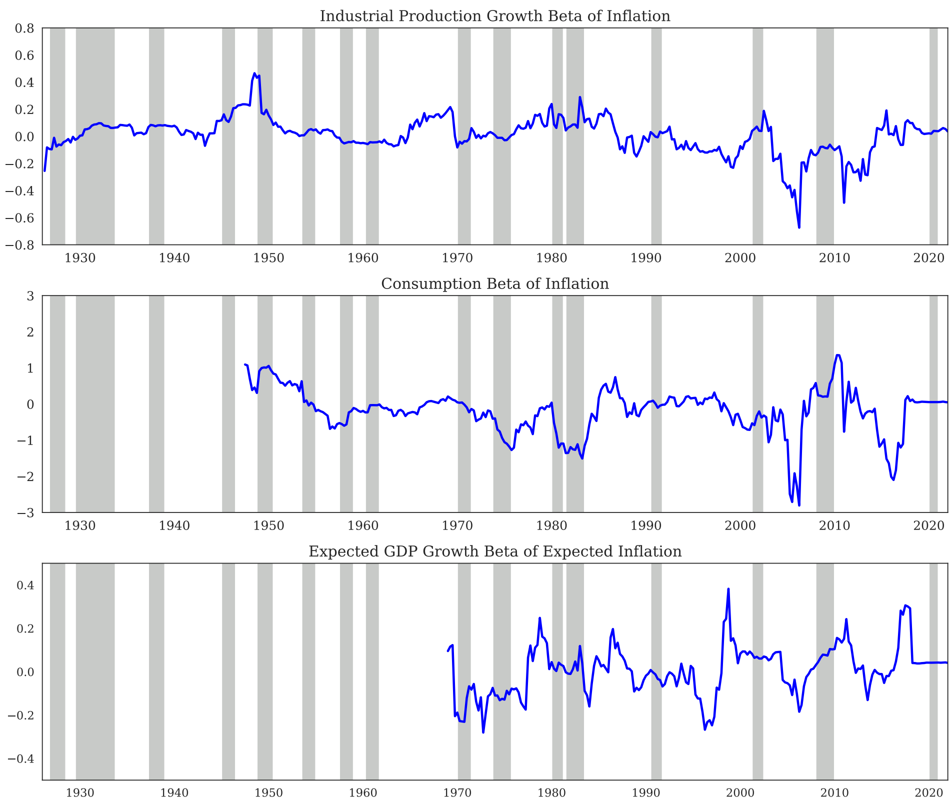


Fig. 3: Consumption Beta of Inflation and Expected Inflation

This figure summarizes the consumption beta of inflation. We use a rolling 5-year window of quarterly data and a first-order VAR for inflation, industrial production growth (consumption growth, GDP growth expectation), and the three-month treasury bill returns to calculate inflation shocks. Beta is defined as $\widehat{Cov_t}\left(\pi_t, x_t\right)/\widehat{Var_t}\left(x_t\right)$. In the first panel, x_t stands for shocks to industrial production growth, ranging from 1926Q1-2021Q4, from the FRED datasets. In the second panel, x_t stands for shocks to consumption growth, ranging from 1947Q1-2021Q4, from BEA. In the last panel, x_t and π_t represent shocks to real GDP growth and GDP deflator expectations from Surveys of Professional Forecasters for the period 1968Q4 - 2021Q4. Shaded areas denote NBER recessions.



Appendices

A Moment-generating Function of Gamma Distribution

We use $\Psi(u)$ to denote the moment-generating functions for the Gamma distributed shocks in volatility states:

$$\Psi(u) = E\left(e^{u\omega_{\sigma,t+1}}\right).$$

For our parameterization of Gamma distribution, the expression for the moment-generating functions is given by:

$$\Psi(u) = \left(1 - \widetilde{\theta} u\right)^{-\widetilde{k}}, \text{ for } u < \frac{1}{\theta},$$

in which:

$$\begin{aligned} \widetilde{k} &= \left[\frac{\sigma_c^2 (1-\nu)}{\sigma_{c\omega}}\right]^2, \\ \widetilde{\theta} &= \frac{\sigma_{c\omega}^2}{\sigma_c^2 (1-\nu_\sigma)}. \end{aligned}$$

It is important to note that even though the volatility shocks are non-Guassian, the model specification belongs to the exponentially affine class. Indeed, the expectations for the exponential of the state variable are exponentially linear in the current states, which generally facilitates the solution for our model.

B Model Solutions of Within-regime LRR Model

The dynamics of consumption growth and inflation are characterized as:

$$
\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \tau_z z_t + \sigma_t \varepsilon_{c,t+1}, \\
\pi_{t+1} &= \mu_\pi + \tau_x x_t + z_t + \varphi_\pi \sigma_t \varepsilon_{\pi,t+1}, \\
x_{t+1} &= \nu_x x_t + \varphi_x \sigma_t \varepsilon_{x,t+1} + \rho_{xz} \varphi_z \sigma_t \varepsilon_{z,t+1}, \\
z_{t+1} &= \nu_z z_t + \rho_{zx} \varphi_x \sigma_t \varepsilon_{x,t+1} + \varphi_z \sigma_t \varepsilon_{z,t+1}, \\
\sigma_{t+1}^2 &= \sigma_c^2 (1 - \nu_\sigma) + \nu_\sigma \sigma_t^2 + \omega_{\sigma,t+1}.
\end{aligned}
$$

The short-run consumption and inflation shocks $\varepsilon_{c,t+1}$ and $\varepsilon_{\pi,t+1}$, as well as the long-run consumption and inflation shocks $\varepsilon_{x,t+1}$ and $\varepsilon_{z,t+1}$ are standard normal. To ensure the positivity of the volatility process, we assume that the volatility shock $\omega_{\sigma,t+1}$ follows a demeaned Gamma distribution: $\omega_{\sigma,t+1} = \widetilde{\omega}_{\sigma,t+1} - E\left(\widetilde{\omega}_{\sigma,t+1}\right)$. The Gamma distribution of $\widetilde{\omega}_{\sigma,t+1}$ is characterized by two parameters, so we specify the mean and volatility of the volatility shocks as:

$$
\begin{aligned}
E\left(\widetilde{\omega}_{\sigma,t+1}\right) &= \sigma_c^2 \left(1 - \nu_\sigma\right), \\
Var\left(\widetilde{\omega}_{\sigma,t+1}\right) &= \sigma_{c\omega}^2.
\end{aligned}
$$

Using the Euler equation for the consumption asset, we obtain that the equilibrium log wealth-to-consumption ratio v_t is linear in the states of the economy:

$$
v_t = A_0 + A_x x_t + A_z z_t + A_\sigma \sigma_t^2
$$

Using the Euler equation (3) and the assumed dynamics of consumption growth and inflation, we derive the solutions coefficients A_x, A_z and A_σ as:

$$\begin{aligned}
A_x &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\nu_x} \\
A_z &= \frac{\left(1-\frac{1}{\psi}\right)\tau_z}{1-\kappa_1\nu_z} \\
A_\sigma &= \frac{\theta\left[\left(1-\frac{1}{\psi}\right)^2+\kappa_1^2\left\{\left(A_x+A_z\rho_{zx}\right)^2\varphi_x^2+\left(A_x\rho_{xz}+A_z\right)^2\varphi_z^2\right\}\right]}{2(1-\kappa_1\nu_\sigma)} \\
A_0 &= \frac{\theta\log\delta+\theta\left(1-\frac{1}{\psi}\right)\mu_c+\theta\kappa_0+\log\Psi\left(\theta\kappa_1A_\sigma\right)}{\theta(1-\kappa_1)}
\end{aligned}$$

Using the equilibrium solution to the wealth-to-consumption ratio, we can write the expression for the real discount factor as follows:

$$\begin{aligned}
m_{t+1} &= m_0+m_xx_t+m_zz_t+m_\sigma\sigma_t^2 \\
&\quad -\lambda_c\sigma_t\varepsilon_{c,t+1}-\lambda_x\varphi_x\sigma_t\varepsilon_{c,t+1}-\lambda_z\varphi_z\sigma_t\varepsilon_{z,t+1}-\lambda_\omega\omega_{\sigma,t+1}.
\end{aligned}$$

The solutions to the discount factor loadings are given by:

$$\begin{aligned}
m_0 &= \theta\log\delta-\gamma\mu_c+(\theta-1)\kappa_0+(\theta-1)A_0(\kappa_1-1)+(\theta-1)\kappa_1A_\sigma\sigma_c^2(1-\nu_\sigma) \\
m_x &= -\frac{1}{\psi}, \\
m_z &= -\frac{\tau_z}{\psi}, \\
m_\sigma &= (\theta-1)A_\sigma(\kappa_1\nu_\sigma-1).
\end{aligned}$$

The market prices of systematic risks can be expressed in terms of underlying preferences and parameters that govern the evolution of consumption growth and inflation:

$$
\begin{aligned}
\lambda_c &= \gamma, \\
\lambda_x &= (1-\theta)\kappa_1 \left(A_z \rho_{zx} + A_x\right), \\
\lambda_z &= (1-\theta)\kappa_1 \left(A_x \rho_{xz} + A_z\right), \\
\lambda_\omega &= (1-\theta)\kappa_1 A_\sigma.
\end{aligned}
$$

We denote $q_{t,n}$ and $q_{t,n}^{\$}$ as the equilibrium solutions to the real and nominal n-period bond prices, respectively. Using the Euler equation, the equilibrium solutions to the real bond prices are affine in state variables:

$$
q_{t,n} = -B_{0,n} - B_{x,n} x_t - B_{z,n} z_t - B_{\sigma,n} \sigma_t^2,
$$

in which the loadings satisfy the recursions:

$$
\begin{aligned}
B_{0,n} &= B_{0,n-1} - m_0 - \log \Psi \left(-\lambda_\omega - B_{\sigma,n-1}\right) + B_{\sigma,n-1} \sigma_c^2 \left(1-\nu\right), \\
B_{x,n} &= B_{x,n-1} \nu_x - m_x, \\
B_{z,n} &= B_{z,n-1} \nu_z - m_z, \\
B_{\sigma,n} &= B_{\sigma,n-1} \nu_\sigma - m_\sigma - \frac{1}{2} \left(\lambda_x + B_{x,n-1} + B_{z,n-1} \rho_{zx}\right)^2 \varphi_x^2 \\
&\quad - \frac{1}{2} \left(\lambda_z + B_{x,n-1} \rho_{xz} + B_{z,n-1}\right)^2 \varphi_z^2 - \frac{1}{2} \lambda_c^2.
\end{aligned}
$$

We define the holding period return of bond as $rb_{t+1}^{(n)} = q_{t+1,n-1} - q_{t,n}$ and obtain:

$$
\begin{aligned}
rb_{t+1}^{(n)} &= G_{0,n} + G_{x,n} x_t + G_{z,n} z_t + G_{\sigma,n} \sigma_t^2 \\
&\quad + \beta_{x,n}^b \varphi_x \sigma_t \varepsilon_{c,t+1} + \beta_{z,n}^b \varphi_z \sigma_t \varepsilon_{z,t+1} + \beta_{\sigma,n}^b \omega_{\sigma,t+1}.
\end{aligned}
$$

in which:

$$G_{0,n} = B_{0,n} - B_{0,n-1} - B_{\sigma,n-1}\sigma_c^2(1-\nu),$$
$$G_{x,n} = B_{x,n} - B_{x,n-1}\nu_x$$
$$G_{z,n} = B_{z,n} - B_{z,n-1}\nu_z$$
$$G_{\sigma,n} = B_{\sigma,n} - B_{\sigma,n-1}\nu_\sigma$$

and the betas are:

$$\beta^b_{x,n} = -(B_{x,n-1} + B_{z,n-1}\rho_{zx})$$
$$\beta^b_{z,n} = -(B_{x,n-1}\rho_{xz} + B_{z,n-1})$$
$$\beta^b_{\sigma,n} = -B_{\sigma,n-1}$$

The risk premium is:

$$\begin{aligned} rp_c &= -Cov\left(m_{t+1}, rb^{(n)}_{t+1}|I_t\right) \\ &= -\left[(B_{x,n-1} + B_{z,n-1}\rho_{zx})\lambda_x\varphi_x^2 + (B_{x,n-1}\rho_{xz} + B_{z,n-1})\lambda_z\varphi_z^2\right]\sigma_t^2 - B_{\sigma,n-1}\lambda_\sigma\sigma^2_{c\omega} \end{aligned}$$

The discounts factor used to price the nominal payoff is given by:

$$\begin{aligned} m^\$_{t+1} &= m_{t+1} - \pi_{t+1} \\ &= m^\$_0 + m^\$_x x_t + m^\$_z z_t + m_\sigma\sigma_t^2 \\ &\quad -\lambda_c\sigma_t\varepsilon_{c,t+1} - \lambda_x\varphi_x\sigma_t\varepsilon_{c,t+1} - \lambda_z\varphi_z\sigma_t\varepsilon_{z,t+1} - \lambda_\omega\omega_{\sigma,t+1} - \lambda^\$_\pi\sigma_t\varepsilon_{\pi,t+1}. \end{aligned}$$

The solutions to the discount factor loadings are given by:

$$
\begin{aligned}
m_0^{\$} &= m_0 - \mu_\pi \\
m_x^{\$} &= -\frac{1}{\psi} - \tau_x, \\
m_z^{\$} &= -\frac{\tau_z}{\psi} - 1, \\
\lambda_\pi^{\$} &= \varphi_\pi.
\end{aligned}
$$

Similarly, when we use the equilibrium solution to the nominal discount factor and the Euler equation, the nominal bond prices are affine in state variables:

$$q_{t,n}^{\$} = -B_{0,n}^{\$} - B_{x,n}^{\$} x_t - B_{z,n}^{\$} z_t - B_{\sigma,n}^{\$} \sigma_t^2,$$

in which the nominal bond loadings satisfy the recursions:

$$
\begin{aligned}
B_{0,n}^{\$} &= B_{0,n-1}^{\$} - m_0^{\$} - \log \Psi\left(-\lambda_\omega - B_{\sigma,n-1}^{\$}\right) + B_{\sigma,n-1}^{\$} \sigma_c^2 (1-\nu), \\
B_{x,n}^{\$} &= B_{x,n-1}^{\$} \nu_x - m_x^{\$}, \\
B_{z,n}^{\$} &= B_{z,n-1}^{\$} \nu_z - m_z^{\$}, \\
B_{\sigma,n}^{\$} &= B_{\sigma,n-1}^{\$} \nu_\sigma - m_\sigma - \frac{1}{2}\left(\lambda_x + B_{x,n-1}\varphi_x + B_{z,n-1}\rho_{xz}\right)^2 \\
&\quad -\frac{1}{2}\left(\lambda_z + B_{x,n-1}\rho_{zx} + B_{z,n-1}\varphi_x\right)^2 - \frac{1}{2}\lambda_c^2 - \frac{1}{2}(\lambda_\pi^{\$})^2.
\end{aligned}
$$

The holding period return of the nominal bond $rb_{t+1}^{\$(n)} = q_{t+1,n-1}^{\$} - q_{t,n}^{\$}$ has a similar form of loadings on state variables and shocks to the real bond except that we replace B coefficients with $B^{\$}$.

C Solutions to LRR Model with Regime Switching

C.1 Price-consumption Ratio

We conjecture that the log price-to-consumption ratio $v_t\left(j\right)$ is linear in the states of the economy:

$$v_t\left(j\right) = A_0(j) + A_1(j)'Y_t.$$

From Campbell-Shiller decomposition, we obtain:

$$r_{c,t+1}\left(j,k\right) = \kappa_0\left(k\right) + \kappa_1\left(k\right) v_{c,t+1}(k) - v_{c,t}\left(j\right) + \Delta c_{t+1}(j).$$

Then, given $s_t = j$ and $s_{t+1} = k$, we obtain:

$$m_{t+1} + r_{c,t+1} = B_0(j,k) + B_1(j,k)'Y_t + B_2(k)'\left(G_t\left(j\right)\varepsilon_{t+1} + \omega_{t+1}^{j}\right),$$

in which:

$$\begin{aligned}
B_0(j,k) &= \theta\log\delta + \theta\left(1-\frac{1}{\varphi}\right)e_1'\mu(j) + \theta k_0(k) + \theta\kappa_1(k)\left[A_0(k) + A_1\left(k\right)'\mu(j)\right] - \theta A_0(j) \\
B_1(j,k) &= \theta\left[\left(1-\frac{1}{\varphi}\right)F\left(j\right)'e_1 + \kappa_1(k)F\left(j\right)'A_1(k) - A_1(j)\right] \\
B_2(k) &= \theta\left[\left(1-\frac{1}{\varphi}\right)e_1 + \kappa_1(k)A_1(k)\right].
\end{aligned}$$

$A_0(j)$ and $A_1(j)$ can be jointly determined by the following equations:

$$\begin{aligned} 1 &= E_t\left[\exp(m_{t+1}+r_{c,t+1})\right] \\ &= \sum_{k=1}^{S}\pi^{jk}E\left[\exp\left\{B_0(j,k)+B_1(j,k)'Y_t+B_2(k)'\left(G_t\left(j\right)\varepsilon_{t+1}+\omega_{t+1}^{j}\right)\right\}|I_t,s_{t+1}\right] \\ &= \sum_{k=1}^{S}\pi^{jk}\left[\exp\left(\begin{array}{c} B_0(j,k)+\log\Psi\left[B_2(k)'e_5\right]-B_2(k)'e_5e_5'\mu(j) \\ +\left\{B_1(j,k)+\frac{1}{2}B_2(k)'H_\sigma\left(j\right)'B_2(k)e_5\right\}'Y_t \end{array}\right)\right] \end{aligned}$$

for $j=1,2,...S$. We log-linearize the above equation around the unconditional mean $\mu\left(j\right)$ as:

$$0=\log\left[g_0(j)\right]-\frac{g_1(j)'}{g_0(j)}\mu\left(j\right)+\frac{g_1(j)'}{g_0(j)}Y_t,$$

in which:

$$\begin{aligned} g_0(j) &= \sum_{k=1}^{S}\pi^{jk}\left\{exp\left(\begin{array}{c} B_0(j,k)+\log\Psi\left[B_2(k)'e_5\right]-B_2(k)'e_5e_5'\mu(j) \\ +\left\{B_1(j,k)+\frac{1}{2}B_2(k)'H_\sigma\left(j\right)'B_2(k)e_5\right\}'\mu(j) \end{array}\right)\right\} \\ g_1(j) &= \sum_{k=1}^{S}\pi^{jk}\left\{\begin{array}{c} \exp\left(\begin{array}{c} B_0(j,k)+\log\Psi\left[B_2(k)'e_5\right]-B_2(k)'e_5e_5'\mu(j) \\ +\left\{B_1(j,k)+\frac{1}{2}B_2(k)'H_\sigma\left(j\right)'B_2(k)e_5\right\}'\mu(j) \end{array}\right) \\ \times\left[B_1(j,k)+\frac{1}{2}B_2(k)'H_\sigma\left(j\right)B_2(k)e_5\right] \end{array}\right\}. \end{aligned}$$

Since the above equation holds for all j, we should obtain:

$$\begin{aligned} \log\left[g_0(j)\right]-\frac{g_1(j)'}{g_0(j)}\mu\left(j\right) &= 0, \\ \frac{g_1(j)}{g_0(j)} &= 0_{6\times 1}. \end{aligned}$$

for $j=1,2,...S$. With $(7\times S)$ equations, we can determine $(7\times S)$ unknowns jointly: $A_0(j)$ and $A_1(j)$ for $j=1,2,...S$.

C.2 Discount Factor

The equilibrium discount factor can be written as:

$$m_{t+1}(j,k) = m_0(j,k) + m_1(j,k)'Y_t - \Lambda(k)' \left(G_t\,(j)\,\varepsilon_{t+1} + \omega_{t+1}^j\right).$$

in which:

$$\begin{aligned} m_0(j,k) &= \theta \log\delta - \gamma e_1'\mu(j) + (\theta-1)\left[\kappa_0(k) + \kappa_1(k)\left\{A_0(k) + A_1(k)'\mu\,(j)\right\} - A_0(j)\right], \\ m_1(j,k) &= -\gamma F(j)'e_1 + (\theta-1)\left[\kappa_1(k)F(j)'A_1(k) - A_1(j)\right]. \end{aligned}$$

and the market prices of risks are:

$$\Lambda(k) = \gamma e_1 + (1-\theta)\,\kappa_1(k)A_1(k).$$

C.3 Return to Consumption Claims

We can express the return on consumption claims as:

$$\begin{aligned} r_{c,t+1}\,(j,k) &= \kappa_0\,(k) + \kappa_1\,(k)\,v_{c,t+1}(k) - v_{c,t}\,(j) + \Delta c_{t+1}(j) \\ &= J_0(j,k) + J_1'(j,k)x_t + \beta_c(k)'\left(G_t\,(j)\,\varepsilon_{t+1} + \omega_{t+1}^j\right), \end{aligned}$$

in which:

$$\begin{aligned} J_0(j,k) &= e_1'\mu(j) + \kappa_0(k) + \kappa_1(k)\left[A_0(k) + A_1(k)'\mu\,(j)\right] - A_0(j), \\ J_1(j,k) &= F(j)'e_1 + \kappa_1(k)F(j)'A_1(k) - A_1(j), \\ \beta_c(k) &= e_1 + \kappa_1(k)A_1(k). \end{aligned}$$

C.4 Return on Equity

We conjecture that the log price-to-dividend ratio $v_{m,t}(j)$ is linear in the states of the economy:

$$v_{m,t}(j) = A_{0,m}(j) + A_{1,m}(j)'Y_t.$$

From Campbell-Shiller decomposition, we obtain:

$$r_{m,t+1}(j,k) = \kappa_{0,m}(k) + \kappa_{1,m}(k)v_{m,t+1}(k) - v_{m,t}(j) + \Delta d_{t+1}(j).$$

Then, given $s_t = j$ and $s_t = k$, we obtain:

$$m_{t+1} + r_{m,t+1} = B_{0,m}(j,k) + B_{1,m}(j,k)'Y_t + B_{2,m}(k)'\left(G_t(j)\,\varepsilon_{t+1} + \omega^j_{t+1}\right)$$

in which:

$$\begin{aligned}
B_{0,m}(j,k) &= m_0(j,k) + \kappa_{0,m}(k) + \kappa_{1,m}(k)\left[A_{0,m}(k) + A_{1,m}(k)'\mu(j)\right] - A_{0,m}(j) + e_6'\mu(s_t) \\
B_{1,m}(j,k) &= F(j)'e_6 + m_1(j,k) + \kappa_{1,m}(k)F(j)'A_{1,m}(k) - A_{1,m}(j) \\
B_{2,m}(k) &= e_6 + k_{1,m}(k)A_{1,m}(k) - \Lambda(k)
\end{aligned}$$

$A_{0,m}(j)$ and $A_{1,m}(j)$ are jointly determined by the following equations:

$$\begin{aligned}
1 &= E_t\left[\exp(m_{t+1} + r_{m,t+1})\right] \\
&= \sum_{k=1}^{S}\pi^{jk}E\left[\exp\left\{B_{0,m}(j,k) + B_{1,m}(j,k)'Y_t + B_{2,m}(k)'\left(G_t(j)\,\varepsilon_{t+1} + \omega^j_{t+1}\right)\right\}|I_t, s_{t+1}\right] \\
&= \sum_{k=1}^{S}\pi^{jk}\left[\exp\left(\begin{array}{c} B_{0,m}(j,k) + \log\Psi\left[B_{2,m}(k)'e_5\right] - B_{2,m}(k)'e_5e_5'\mu(j) \\ + \left\{B_{1,m}(j,k) + \frac{1}{2}B_{2,m}(k)'H_\sigma(j)\,B_{2,m}(k)e_5\right\}'Y_t \end{array}\right)\right]
\end{aligned}$$

for $j=1,2,...S$. We log-linearize the above equation around the unconditional mean $\mu(j)$ as:

$$0=\log\left[g_{0,m}(j)\right]-\frac{g_{1,m}(j)'}{g_{0,m}(j)}\mu(j)+\frac{g_{1,m}(j)'}{g_{0,m}(j)}Y_t,$$

in which:

$$\begin{aligned}
g_{0,m}(j) &= \sum_{k=1}^{S}\pi^{jk}\left\{\exp\left(\begin{array}{c} B_{0,m}(j,k)+\log\Psi\left[B_{2,m}(k)'e_5\right]-B_{2,m}(k)'e_5e_5'\mu(j) \\ +\left\{B_{1,m}(j,k)+\frac{1}{2}B_{2,m}(k)'H_\sigma(j)'B_{2,m}(k)e_5\right\}'\mu(j)\end{array}\right)\right\} \\
g_{1,m}(j) &= \sum_{k=1}^{S}\pi^{jk}\left\{\begin{array}{c}\exp\left(\begin{array}{c} B_{0,m}(j,k)+\log\Psi\left[B_{2,m}(k)'e_5\right]-B_{2,m}(k)'e_5e_5'\mu(j) \\ +\left\{B_{1,m}(j,k)+\frac{1}{2}B_{2,m}(k)'H_\sigma(j)B_{2,m}(k)e_5\right\}'\mu(j)\end{array}\right) \\ \times\left[B_{1,m}(j,k)+\frac{1}{2}B_{2,m}(k)'H_\sigma(j)B_{2,m}(k)e_5\right]\end{array}\right\}
\end{aligned}$$

Since these equation hold for all j ($j=1,2,...S$), we should obtain:

$$\begin{aligned}
\log\left[g_{0,m}(j)\right]-\frac{g_{1,m}(j)'}{g_{0,m}(j)}\mu(j) &= 0, \\
\frac{g_{1,m}(j)}{g_{0,m}(j)} &= 0_{6\times 1}.
\end{aligned}$$

for $j=1,2,...S$. With $(7\times S)$ equations, we can determine $(7\times S)$ unknowns jointly: $A_{0,m}(j)$ and $A_{1,m}(j)$ for $j=1,2,...S$.

We can express the return on equity as:

$$\begin{aligned}
r_{m,t+1}(j,k) &= \kappa_{0,m}(k)+\kappa_{1,m}(k)v_{m,t+1}(k)-v_{m,t}(j)+\Delta d_{t+1}(j) \\
&= J_{0,m}(j,k)+J_{1,m}(j,k)'Y_t+\beta_d(k)'\left(G_t(j)\varepsilon_{t+1}+\omega_{t+1}^{j}\right)
\end{aligned}$$

in which:

$$
\begin{aligned}
J_{0,m}(j,k) &= e_6'\mu(j) + \kappa_{0,m}(k) + \kappa_{1,m}(k)\left[A_{0,m}(k) + A_{1,m}(k)'\mu\left(j\right)\right] - A_{0,m}(j), \\
J_{1,m}(j,k) &= F(j)'e_6 + \kappa_{1,m}(k)F(j)'A_{1,m}(k) - A_{1,m}(j), \\
\beta_d(k) &= e_6 + \kappa_{1,m}(k)A_{1,m}(k).
\end{aligned}
$$

C.5 Real Bond Prices

The log prices at period t of real discount bonds with n periods to maturity, $q_{t,n}$, satisfy the Euler equation:

$$
\exp(q_{t,n}) = E_t \exp(m_{t+1} + q_{t+1,n-1}).
$$

The log bond price also has an affine structure of the states of the economy:

$$
q_{t,n}(j) = B_{0,n}(j) + B_{1,n}(j)'Y_t
$$

The coefficients $B_{0,n}$ and $B_{1,n}$ are state s_t dependent, and satisfy the following recursive relations:

$$
\begin{aligned}
B_{0,n}(j) &= \log\left[g_{0,b}(j)\right] - \frac{g_{1,b}(j)'}{g_{0,b}(j)}\mu\left(j\right), \\
B_{1,n}(j) &= \frac{g_{1,b}(j)}{g_{0,b}(j)}.
\end{aligned}
$$

in which:

$$
\begin{aligned}
g_{0,b}(j) &= \sum_{k=1}^{S} \pi^{jk} \left\{ \exp \left(\begin{array}{c} D_0(j,k) + \log \Psi \left[D_2(k)' e_5 \right] - D_2(k)' e_5 e_5' \mu(j) \\ + \left\{ D_1(j,k) + \frac{1}{2} D_2(k)' H_\sigma(j) D_2(k) e_5 \right\}' \mu(j) \end{array} \right) \right\} \\
g_{1,b}(j) &= \sum_{k=1}^{S} \pi^{jk} \left\{ \begin{array}{c} \exp \left(\begin{array}{c} D_0(j,k) + \log \Psi \left[D_2(k)' e_5 \right] - D_2(k)' e_5 e_5' \mu(j) \\ + \left\{ D_1(j,k) + \frac{1}{2} D_2(k)' H_\sigma(j) D_2(k) e_5 \right\}' \mu(j) \end{array} \right) \\ \times \left[D_1(j,k) + \frac{1}{2} D_2(k)' H_\sigma(j) D_2(k) e_5 \right] \end{array} \right\} \\
D_0(j,k) &= m_0(j,k) - B_{0,n-1}(k) - B_{1,n-1}(k)' \mu(j), \\
D_1(j,k) &= m_1(j,k) - F(j)' B_{1,n-1}(k), \\
D_2(k) &= -\left[\Lambda(k) + B_{1,n-1}(k) \right],
\end{aligned}
$$

with $B_{0,n}(j) = B_{1,n}(j) = 0$ (real bond that matures now has a price of one).

Wed define the holding period return of bonds as $rb_{t+1}^{(n)} = q_{t+1}^{(n-1)} - q_t^{(n)}$, and thus:

$$
rb_{t+1}^{(n)}(j,k) = G_{0,n}(j,k) + G_{1,n}(j,k)' Y_t + \beta_{b,n}(j,k)' \left(G_t(j) \varepsilon_{t+1} + \omega_{t+1}^j \right),
$$

in which

$$
\begin{aligned}
G_{0,n}(j,k) &= B_{0,n}(j) - B_{0,n-1}(k) - B_{1,n-1}(k)' \mu(j), \\
G_{1,n}(j,k) &= B_{1,n}(j) - F(j)' B_{1,n-1}(k s_{t+1}), \\
\beta_{b,n}(j,k) &= -B_{1,n-1}(k).
\end{aligned}
$$

C.6 Nominal Discount Factor

The equilibrium stochastic discount factor can be written as follows:

$$
m_{t+1}^{\$} = m_{t+1} - \pi_{t+1} = m_0^{\$}(j,k) + m_1^{\$}(j,k)' x_t - \Lambda^{\$}(j,k)' \left(G_t(j) \varepsilon_{t+1} + \omega_{t+1}^j \right),
$$

in which:

$$\begin{aligned} m_0^{\$}(j,k) &= m_0(j,k) - e_2'\mu(j), \\ m_1^{\$}(j,k) &= m_1(j,k) - F(j)'e_2, \\ \Lambda^{\$}(j,k) &= \Lambda(k) + e_2. \end{aligned}$$

The nominal bond pricing and nominal bond risk premium work exactly the same as the real bond case. We can use the same recursion to compute nominal bond price except that we replace the real SDF, m_{t+1}, with the nominal SDF, $m_{t+1}^{\$}$.