PHBS WORKING PAPER SERIES

On the Foundations of Competitive Search Equilibrium with and without Market Makers

James Albrecht
Georgetown University and IZA

Xiaoming Cai
Peking University

Pieter Gautier
Vrije Universiteit Amsterdam
and Tinbergen Institute

Susan Vroman
Georgetown University and IZA

September 2021

Working Paper 20210905

Abstract

The literature offers two foundations for competitive search equilibrium, a Nash approach and a market-maker approach. When each buyer visits only one seller (or each worker makes only one job application), the two approaches are equivalent. However, when each buyer visits multiple sellers, this equivalence can break down. Our paper analyzes competitive search equilibrium with simultaneous search using the two approaches. We consider four cases defined by (i) the surplus structure (are the goods substitutes or complements?) and (ii) the mechanism space (do sellers post fees or prices?). With fees, the two approaches yield the same constrained efficient equilibrium. With prices, the equilibrium allocation is the same using both approaches if the goods are complements, but is not constrained efficient. In the case in which only prices are posted and the goods are substitutes, the equilibrium allocations from the two approaches are different.

Keywords: Multiple application, competitive search, market makers, efficiency
JEL Classification: C78, D44, D83

Peking University HSBC Business School
University Town, Nanshan District
Shenzhen 518055, China

On the Foundations of Competitive Search Equilibrium with and without Market Makers

James Albrecht* Xiaoming Cai† Pieter Gautier‡ Susan Vroman§

June 30, 2021

Abstract

The literature offers two foundations for competitive search equilibrium, a Nash approach and a market-maker approach. When each buyer visits only one seller (or each worker makes only one job application), the two approaches are equivalent. However, when each buyer visits multiple sellers, this equivalence can break down. Our paper analyzes competitive search equilibrium with simultaneous search using the two approaches. We consider four cases defined by (i) the surplus structure (are the goods substitutes or complements?) and (ii) the mechanism space (do sellers post fees or prices?). With fees, the two approaches yield the same constrained efficient equilibrium. With prices, the equilibrium allocation is the same using both approaches if the goods are complements, but is not constrained efficient. In the case in which only prices are posted and the goods are substitutes, the equilibrium allocations from the two approaches are different.

JEL classification: C78, D44, D83.

Keywords: multiple applications, competitive search, market makers, efficiency.

*Georgetown University and IZA. email: albrecht@georgetown.edu.

†Peking University HSBC Business School. email: xmingcai@gmail.com.

‡Vrije Universiteit Amsterdam and Tinbergen Institute. email: p.a.gautier@vu.nl.

§Georgetown University and IZA. email: susan.vroman@georgetown.edu.

1 Introduction

In competitive search models, capacity-constrained agents on one side of the market post and commit to terms of trade. Agents on the other side of the market, after observing all posted terms of trade, decide where to direct their search. Consider, for example, a product market application of competitive search with S sellers, each with one unit of a homogenous good to sell, and B buyers, each wanting to purchase one unit of the good. Each seller posts a price, and buyers, after observing all posted prices, direct their search. That is, each buyer chooses a seller from whom he or she tries to purchase the good. When setting its price, each seller faces a tradeoff. The higher the price, the greater is the payoff the seller receives if the good is sold but the lower is the probability of sale. The seller chooses a price to maximize his or her expected payoff taking this tradeoff into account through a market utility constraint; i.e., the seller realizes that any buyers its posted price might attract must expect a payoff no lower than is available elsewhere in the market. In competitive search equilibrium, each seller takes the buyer expected payoff that is available elsewhere in the market, i.e., the "market utility," as given. This, of course, requires B and S sufficiently large, i.e., competitive search equilibrium is a large-market concept.

The literature offers two common interpretations of competitive search equilibrium. The first views competitive search equilibrium as the limit of a sequence of Nash equilibria as the numbers of players on the two sides of a market get arbitrarily large. Consider another standard example, a labor market in which v firms, each with one vacancy, post and commit to wages; then u unemployed workers direct their search after observing all posted wages. For given u and v, one can compute the symmetric Nash equilibrium of this game. Letting $v, u \to \infty$ while holding $\lambda = u/v$ fixed, we can compute the sequence of Nash equilibria and take the limit. This gives the competitive search equilibrium. This first interpretation is the one presented in Peters (2000), Burdett et al. (2001) , Galenianos and Kircher (2012) and Albrecht et al. (2020) among others.

A second interpretation of competitive search equilibrium uses the concept of "market maker." In the labor market example, again imagine an arbitrarily large market, i.e., $v, u \to \infty$ with λ fixed. Suppose all firms post the same wage, w, so that workers randomize their search and each vacancy expects λ job seekers. Then (w, λ) is a symmetric competitive search equilibrium if there is no profitable possibility for a market maker to set up a "submarket" in which firms are promised an arrival rate $\widetilde{\lambda}$ of job seekers in return for posting a wage of $\widetilde{w}$. This is the interpretation of competitive search equilibrium presented in Moen (1997) and Mortensen and Wright (2002) among others.

In the product market example when each buyer can contact only one seller and in the labor market example when each worker can contact only one firm, the above interpretations lead to the same equilibrium allocation. This can be shown by comparing the analyses of, e.g., Burdett et al. (2001) and Mortensen and Wright (2002). In the labor market example, when each worker can only contact one firm, a deviating firm and a deviating market maker offer workers the same alternative, namely, the chance to apply to a firm that is offering a wage that differs from the one that other firms are posting. However, in settings in which workers make multiple applications, the equivalence can break down. A deviating market maker can attract multiple firms giving workers the option to send more than one application to the deviant submarket, but under the Nash interpretation, each worker can send at most one application to the deviant firm. This difference can change the equilibrium outcome. Similarly, in a product market in which each buyer can approach multiple sellers, the Nash and market maker interpretations of competitive search equilibrium can lead to different outcomes.

Galenianos and Kircher (2009) consider a third way to calculate competitive search equilibria. To evaluate the expected payoff from a deviation by a seller, they let a measure zero of the sellers tremble and offer any mechanism from a given mechanism space. This implies that the number of submarkets is the same as the number of elements in the mechanism space. In their equilibrium, no submarket yields a strictly higher expected payoff for sellers than the equilibrium payoff. If a buyer who visits a deviant submarket pays all of his or her other visits to the candidate equilibrium submarkets, then the equilibrium concept of Galenianos and Kircher (2009) with trembling-hand sellers reduces to the market-maker equilibrium, which is indeed the case for the model in Galenianos and Kircher (2009). However, it can be the case that for some other models, a buyer is better off visiting more than one nonequilibrium submarket. In this case, their equilibrium concept differs from the market-maker equilibrium concept. If a seller considers deviating to one submarket, this seller must then believe that some other sellers will deviate to some other submarket simultaneously, even if this is suboptimal for those other sellers. In the market-maker equilibrium, in contrast, a potential deviant seller believes that other sellers will never enter submarkets that give them suboptimal payoffs. We therefore focus on competitive search equilibria based on either single-seller (Nash) deviations or on market-maker deviations.

The purpose of our paper is to examine these two equilibrium concepts in a competitive search environment in which buyers direct their search to multiple sellers or in which workers apply to more than one job. We are interested in determining which market characteristics lead to equivalence, i.e., when do the Nash and market-maker approaches yield the same equilibrium outcomes, and when do they differ? Under what circumstances do the two approaches generate equilibrium outcomes that are constrained efficient? We can think of a market maker as a type of intermediary, so our results also contribute to the literature that studies the role of intermediaries in markets.

	perfect complements	perfect substitutes
entry fees	Nash = MM (= efficiency)	Nash = MM (= efficiency)
prices	Nash = MM ($\neq$ efficiency)	Nash $\neq$ MM (= efficiency)

Table 1: Comparison of the Nash and the market-maker (MM) equilibria

The results of our paper are that under some conditions, the Nash and market-maker equilibria coincide; under other conditions, the two equilibrium allocations differ. When the Nash and market-maker equilibria are the same, the common equilibrium allocation is constrained efficient under some conditions but inefficient under others, and sometimes the common equilibrium allocation is efficient even though the Nash and market-maker out-of-equilibrium payoffs differ. When the Nash and market-maker approaches give different equilibrium allocations, only the market-maker approach gives efficiency. We generate this range of outcomes by (i) allowing for two different mechanisms, namely, entry fees or prices, and (ii) allowing for goods (or, in the labor market, jobs or workers) to be complements or substitutes. This gives four (2×2) cases. Table 1 summarizes the results.

It is noteworthy that the market-maker approach does not always yield the constrained efficient outcome, even though market makers have the potential to attract all visits of buyers who visit the deviant submarket. Suppose that market makers can enforce the following *exclusive participation rule*: buyers should either pay all visits to the submarket or none. In this case, we can think of the market maker as purchasing a "bundle" of buyer visits at a price equal to the buyers' market utility with the objective of maximizing seller profit. Then following the logic of competitive search models with single visits, the equilibrium is always constrained efficient. However, in the absence of such an *exclusive participation rule*, the number of visits per buyer to the deviant submarket is chosen optimally by the buyers and depends on whether the visits are strategic complements or substitutes. If they are strategic substitutes, then buyers who visit the deviant submarket will choose to pay exactly one visit there, and the two approaches (Nash and market-maker) yield the same outcome on and off the equilibrium path. We find that buyer visits are strategic substitutes when only entry fees are employed and the goods are perfect substitutes, but when the mechanism consists of prices alone, the visits are strategic substitutes when the goods are perfect complements. If the visits are strategic complements, then buyers who visit the deviant submarket will choose to pay all their visits there, which implies that the market-maker equilibrium is always constrained efficient. This happens when sellers post only entry fees and the goods are perfect complements and when sellers post only prices and the goods are perfect substitutes. In the entry fee case, the difference between the Nash outcome and the market-maker outcome arises only off equilibrium, but, when only prices are posted, the equilibrium outcomes also differ.

Although there is a large and growing literature on competitive search, only a few papers have considered multiple visits/applications despite its empirical importance.[1] Albrecht et al. (2006), for example, allow workers to apply to multiple firms and show that when firms Bertrand compete for workers with multiple offers, that unlike the case in which workers make only a single application, the competitive search Nash equilibrium is not constrained efficient.[2] Albrecht et al. (2020) examine this further and show that the above conclusion depends on the choice set of firms: if firms are allowed to post both prices and fees, then there exists a continuum of competitive search Nash equilibria, one of which is constrained efficient. As a complement to these results, Albrecht et al. (2020) add an appendix in which they characterize the equilibrium outcome when a measure zero of firms deviates "collectively." In this case, the unique equilibrium is again constrained efficient. Our current paper provides a general foundation for competitive search models in which agents make multiple visits/applications using both the Nash and market-maker approaches.

In the next section, we set up a model of competitive search that allows for simultaneous search by buyers in a product market or multiple job applications by workers. The setup allows firms to compete by posting general mechanisms. Section 3 presents the case in which firms compete by posting fees, and Section 4 presents the case in which firms can only post prices (or wages). The final section contains concluding remarks.

2 The Model

2.1 Setup

There is a continuum of identical buyers with measure B and a continuum of identical sellers with measure S. Each seller has one unit of an indivisible good to sell. Both buyers and sellers are risk neutral. The value of no trade for a seller or a buyer is normalized to zero. Buyers may demand more than one unit of the good. We focus on two polar cases: (i) perfect substitutes where the value of obtaining one unit of the good is (normalized to) one, and the value of any extra unit is zero, which implies that buyers will never buy more than one unit, (ii) perfect complements where the value of $\overline{a}$ (≥ 2) or more offers of the good is normalized to one, and any smaller number of offers has value zero, which implies that buyers will either purchase zero or $\overline{a}$ units of the good. Many examples of complements

[1] See Wright et al. (2021) for an up-to-date review of the competitive search literature.

[2] Galenianos and Kircher (2009) and Kircher (2009) are two other papers that study multiple applications in the labor market, but in their models, the two approaches (Nash and market-maker) are equivalent so the issues that we consider in this paper do not arise.

involve different products, but for simplicity we restrict ourselves to multiple units of the same good that need to be purchased from different suppliers. In the labor market, perfect complements could be multiple jobs needed to make work worthwhile. For example, consider a cleaner for whom it is only worth incurring the travel costs when he or she receives an offer for both a morning and an afternoon job. Alternatively, switching the roles of workers and firms, a startup company might need a certain number of workers to start production. For a product market example, consider a situation in which different ticket agencies only have single tickets left for an opera and a couple wants to go to the opera together or not go at all.

One side of the market (in the product market, this is the seller) posts and commits to a mechanism ω from a mechanism space Ω. After observing all posted mechanisms, buyers direct their search, and buyers can visit at most a ($\geq \overline{a} \geq 2$) sellers. To simplify the analysis, for the case of perfect complements, we assume that $\overline{a} = a$.

A mechanism ω is a list $(t, p_1, \ldots, p_a)$, where t is a fee and p_i is the price if the buyer has received i offers in total. Since all buyers are identical ex ante, sellers must randomly select a buyer (if they have any).[3] We do not allow for recall. That is, if a seller's chosen buyer does not accept the offer, we do not allow the seller to select another buyer. Since we consider a large market, we assume that buyers cannot coordinate their visiting strategies, i.e., they use symmetric strategies. Finally, sellers must independently and simultaneously select a buyer, and trade is conducted according to the posted mechanism. Below we consider two mechanism spaces Ω. Together with the two surplus structures (perfect complements and perfect substitutes), this generates a variety of results that illustrate the relationship between the two interpretations of competitive search equilibrium that are explored below. The first is the simplest case where Ω consists of entry fees only, $\Omega = \{(t, 0, \ldots, 0)\}$ ($p_i = 0$ for $i \geq 1$). In this case, a buyer has to pay an entry fee t to visit a seller; after collecting all the entry fees, sellers will offer the good randomly to one of the buyers for free. The second Ω does not allow fees, i.e., $\Omega = \{(0, p_1, \ldots, p_a)\}$. Note that price posting is a special case in which $p_i = p$ for $i \geq 1$.

We allow for a general constant returns to scale meeting technology. Consider a seller with expected queue length λ; i.e., λ is the expected number of buyers contacting the seller. The probability that the seller meets at least one buyer is $m(\lambda)$, which is assumed to be strictly increasing and strictly concave. We do not assume a particular functional form for the meeting technology $m(\lambda)$. Special cases include the urn ball, $m(\lambda) = 1 - e^{-\lambda}$, and the geometric, $m(\lambda) = \lambda/(1+\lambda)$, both of which are extensively used in the literature.[4] We assume

[3] That is, before a particular buyer is selected, the seller does not observe the number of other offers that its buyers have.

[4] With an urn-ball meeting technology, the probability that a seller meets exactly n buyers is given by

that $\varepsilon_m(\lambda) \equiv \lambda m'(\lambda)/m(\lambda)$, the elasticity of $m(\lambda)$, is strictly decreasing. This implies that $\lim_{\lambda\to 0} \varepsilon_m(\lambda) = 1$ and $\lim_{\lambda\to\infty} \varepsilon_m(\lambda) = 0$ as is standard in the literature.[5]

If all sellers face the same expected queue length, then that common expected queue length is $\lambda = aB/S$, but in general, the expected queue length depends on the posted mechanism and may vary from seller to seller. For notational convenience, we let $h(\lambda)$ denote the probability that a buyer's visit fails to lead to an offer:

$$h(\lambda) = 1 - \frac{m(\lambda)}{\lambda}, \tag{1}$$

where we have used the fact that a buyer's visit leads to an offer with probability $m(\lambda)/\lambda$ since the seller treats all buyers symmetrically. To simplify the exposition, we assume that $\lim_{\lambda\to 0} h(\lambda) = 0$; as the number of buyers goes to zero, each buyer visit leads to an offer. Note that $h(\lambda)$ is strictly increasing since $m(\lambda)$ is strictly increasing and strictly concave. Finally, note that $m(\lambda)$, or equivalently $h(\lambda)$, completely determine the buyers' and sellers' probabilities of trade. They do not depend on the probabilities that a seller meets exactly one buyer, two buyers, and so on, as long as the sum of these probabilities is $m(\lambda)$. We now impose a mild restriction on the meeting technology, namely, that the expected number of buyers who arrive at a seller equals the expected queue length. Eeckhout and Kircher (2010) defined this property as meetings are *nonrival*. Equivalently, we assume that when a buyer contacts a seller, the buyer can always participate in the seller's mechanism. This assumption simplifies the analysis of equilibrium when sellers post fees and holds for common meeting technologies such as the urn-ball.

Finally, we assume that the number of buyers is determined by free entry. We suppose there is a large measure of potential buyers and that each must pay a fixed cost K to participate, where $0 < K < 1$. The measure of sellers in the market is given exogenously. By endogenizing the number of buyers we can investigate whether the market equilibrium is efficient by contrasting equilibrium tightness with the social planner's market tightness.

2.2 Payoffs and equilibrium

To start, we define some terms. Consider a buyer who visits k deviant sellers who all post mechanism $\widetilde{\omega}$ with corresponding expected queue length $\widetilde{\lambda}$ and who visits $a-k$ non-deviant sellers who all post mechanism ω with corresponding expected queue length λ, where $1 \leq$

$e^{-\lambda}\frac{\lambda^n}{n!}$ for $n = 0, 1, \ldots$. With a geometric meeting technology, the corresponding probability is $\frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^n$.

[5] Since $m(0) = 0$, by L'Hôpital's rule we have $\lim_{\lambda\to 0} \varepsilon_m(\lambda) = \lim_{\lambda\to 0} \lambda m'(\lambda)/m(\lambda) = 1$. Since $\varepsilon_m(\lambda)$ is always positive and is assumed to be strictly decreasing, $\lim_{\lambda\to\infty} \varepsilon_m(\lambda)$ exists. If this limit is some $x > 0$, then for λ large enough, we have $\log m(\lambda) = x \log \lambda +$ some constant; the right-hand side tends to infinity when $\lambda \to \infty$. However, since $m(\lambda)$ is always smaller than 1, we have a contradiction and x must be 0.

$k \leq a$. The expected payoff for this buyer is denoted by $U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$. In general, when buyers can visit multiple sellers, a seller's expected payoff depends not only on the seller's own mechanism and expected queue length, but also on which other sellers the buyer visits.[6] Assume that all buyers that a deviant seller attracts pay k visits to deviant sellers who post mechanism $\widetilde{\omega}$ with corresponding expected queue length $\widetilde{\lambda}$ and $a - k$ visits to non-deviant sellers with mechanism ω and expected queue length λ. We denote the expected payoff of a deviant seller by $\pi_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$.

We look for a symmetric, pure-strategy equilibrium in which all sellers post the same mechanism, ω. In this case, buyers simply randomize their visits, and their *equilibrium* expected payoff, or market utility $\overline{U}$, is given by

$$\overline{U} = U_0(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda) = U_k(\omega, \lambda, \omega, \lambda) \text{ for any } k \geq 1, \tag{2}$$

where the second equality follows from the degenerate case $\widetilde{\omega} = \omega$ and $\widetilde{\lambda} = \lambda$. Similarly, the *equilibrium* expected payoff of sellers is given by $\pi_k(\omega, \lambda, \omega, \lambda)$ for any $k \geq 1$.

To establish that a mechanism ω is an equilibrium, we need to show that no profitable deviation exists. However, since buyers can visit multiple sellers simultaneously, single-seller deviations and market-maker deviations, which have been used interchangeably in the literature, may yield different outcomes.

Consider first the equilibrium in which there exists no profitable single-seller deviation. To establish that there is no profitable deviation, consider a deviant seller who posts a mechanism $\widetilde{\omega}$ and expects queue length $\widetilde{\lambda}$. Since only a single seller deviates, buyers who decide to visit the deviant seller must pay their other $a-1$ visits to nondeviant sellers. Since we consider a large market, a deviation by a single seller does not change the market utility of buyers. Thus buyer optimality implies that

$$\overline{U} = U_1(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda), \tag{3}$$

where the left-hand side denotes the buyer value of visiting only non-deviant sellers and the right-hand side denotes the buyer value of visiting the deviant seller and $a - 1$ non-deviant sellers. Equation (3) is the indifference condition that determines the expected queue length at the deviant seller. The expected payoff of the deviant seller is then

$$\pi^d(\widetilde{w}) = \pi_1(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda), \tag{4}$$

[6] As we discuss below, when sellers post only fees, a seller's expected payoff depends only on his or her posted fee and expected queue length (see equation (13) below), and does not depend on which other sellers buyers visit.

where $\widetilde{\lambda}$ is implicitly determined by $\widetilde{\omega}$ through equation (3).

Definition 1. *In a symmetric pure-strategy* ***competitive search Nash equilibrium****, sellers choose a mechanism* ω *and buyers receive their market utility* $\overline{U}$ *such that the following conditions are satisfied.*

1. *Buyer optimality. Market utility* $\overline{U}$ *is given by equation* (2) *on and off the equilibrium path. For buyers who visit the deviant seller, the indifference condition,* (3)*, holds.*
2. *Seller optimality. There exists no profitable deviation for sellers. That is, the expected payoff from a deviation, given by equation* (4)*, does not exceed the equilibrium payoff.*
3. *Free entry. The expected buyer payoff equals the entry cost* K*.*

Next, we consider the market-maker equilibrium. A market maker can open a submarket that is characterized by mechanism $\widetilde{\omega}$. This submarket can potentially attract multiple buyers and multiple sellers, while taking the market utility of buyers as given. The difference relative to the competitive search Nash equilibrium case is that a buyer can visit more than one deviant seller. A market maker opens a deviant submarket in the following way. First, the market maker announces a mechanism $\widetilde{\omega}$ and sellers that are part of the deviant submarket are required to post $\widetilde{\omega}$. In return, the market maker promises market tightness $\widetilde{\lambda}$. The constraint reflects the idea that the market maker has to make it worthwhile for buyers to participate in the deviant submarket. The mechanism ω is a competitive search market-maker equilibrium if no market maker has an incentive to open a submarket with a different mechanism.

Formally, suppose that the deviant submarket has mechanism $\widetilde{\omega}$ and expected queue length $\widetilde{\lambda}$. Then buyer optimality implies that

$$\overline{U} = \max_{1 \leq k \leq a} U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda) \tag{5}$$

where the right-hand side is the payoff of a buyer who optimally chooses k, the number of visits to the deviant submarket.

In all our models below, $U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$ is unsurprisingly strictly decreasing in $\widetilde{\lambda}$ since buyers benefit from shorter queues. To analyze the solution to the buyer's problem (5), consider the indifference condition $\overline{U} = U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$ for a given k, from which we can solve $\widetilde{\lambda}$ as a function of $\widetilde{\omega}$. With a slight abuse of notation, call this solution $\widetilde{\lambda} = \lambda_k(\widetilde{\omega}, \omega, \lambda)$. Lemma 1 below shows that buyer optimality can then be equivalently formulated as a no-arbitrage condition: the optimal k must be such that the expected queue length in the deviant submarket equals the largest possible value.

Lemma 1. *Assume that $U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$ is strictly decreasing in $\widetilde{\lambda}$. Then the number of visits that a buyer who visits the deviant submarket pays to that submarket is the k that solves the following maximization problem,*

$$\max_{1 \leq k \leq a} \lambda_k(\widetilde{\omega}, \omega, \lambda), \tag{6}$$

where $\lambda_k(\widetilde{\omega}, \omega, \lambda)$ is the expected queue length, $\widetilde{\lambda}$, that solves $\overline{U} = U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$.

Proof. To see the logic behind (6), suppose otherwise that k_1 is the equilibrium solution and $\lambda_{k_1}(\widetilde{\omega}, \omega, \lambda) < \lambda_{k_2}(\widetilde{\omega}, \omega, \lambda)$ for some k_2. In that case, the no-arbitrage condition would be violated because a buyer could then pay k_2 visits to the deviant submarket $(\widetilde{w}, \lambda_{k_1}(\widetilde{\omega}, \omega, \lambda))$ and obtain an expected payoff that exceeds the market utility: $\overline{U} = U_{k_2}(\widetilde{\omega}, \lambda_{k_2}(\widetilde{\omega}, \omega, \lambda), \omega, \lambda) < U_{k_2}(\widetilde{\omega}, \lambda_{k_1}(\widetilde{\omega}, \omega, \lambda), \omega, \lambda)$, where the first equality follows from the definition of the function $\lambda_k(\widetilde{\omega}, \omega, \lambda)$, and the second inequality holds because $U_{k_2}(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$ is strictly decreasing in $\widetilde{\lambda}$. □

Compared to the optimization problem in (5), the above alternative formulation is more straightforward and can be understood as follows. For a given mechanism $\widetilde{\omega}$, $\lambda_k(\widetilde{\omega}, \omega, \lambda)$ is the queue that makes buyers indifferent between receiving the market utility and paying k visits to the deviant submarket and the remaining $a - k$ visits to the non-deviant submarket. Whenever there are too few visits to the deviant submarket ($\widetilde{\lambda} < \lambda_k(\widetilde{\omega}, \omega, \lambda)$ for some k), there will exist a profitable deviation and an individual buyer could obtain an expected payoff strictly above their market utility by visiting the deviant submarket. When the longest queue is reached with $k = 1$, this means that buyers who visit the deviant submarket pay exactly one visit there, so there are relatively many buyers visiting the deviant submarket, while if it is reached with $k = a$, it means that buyers who visit the deviant submarket pay all visits there, so there are relatively few buyers visiting the deviant submarket.

As before, the expected payoff of a seller who joins the deviant submarket is

$$\pi^d(\widetilde{w}) = \pi_k(\widetilde{\omega}, \lambda_k(\widetilde{\omega}, \omega, \lambda), \omega, \lambda) \tag{7}$$

where k solves the buyer's maximization problem in (5) and $\lambda_k(\widetilde{\omega}, \omega, \lambda)$ is the value of $\widetilde{\lambda}$ that solves equation (5).

Definition 2. *In a symmetric pure-strategy* ***competitive search market-maker equilibrium****, sellers choose a mechanism ω and buyers receive their market utility $\overline{U}$ such that the following conditions are satisfied.*

1. *Buyer optimality. Market utility $\overline{U}$ is given by equation (2) on and off the equilibrium*

path. For buyers who apply to the deviant submarket, the indifference condition, (5), *holds.*

2. *Seller and market maker optimality. No market maker can create a profitable submarket. That is, the expected payoff of sellers in a deviant submarket, which is given by equation* (7)*, is no greater than their equilibrium payoff.*

3. *Free entry. The expected buyer payoff equals the entry cost K.*

In all the models that we consider below, the mechanisms are characterized by a scalar (ω is either a price or a fee), and $U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$ is strictly decreasing in $\widetilde{\omega}$. Thus from the buyer indifference condition $\overline{U} = U_k(\widetilde{\omega}, \widetilde{\lambda}, \omega, \lambda)$, we can solve $\widetilde{\omega}$ as a function of $\widetilde{\lambda}$, which, as we will see later, is easier to analyze than the functions $\lambda_k(\widetilde{\omega}, \omega, \lambda)$. With a slight abuse of notation, denote this solution by $\widetilde{\omega} = \omega_k(\widetilde{\lambda}, \omega, \lambda)$. That is, for a given expected queue of buyers in a deviant submarket, $\widetilde{\lambda}$, there exists a price or fee $\omega_k(\widetilde{\lambda}, \omega, \lambda)$ such that a given buyer is indifferent between receiving his or her market utility on the one hand and paying k visits to the deviant submarket and the remaining $a - k$ visits to the non-deviant submarket on the other hand. Buyer optimality can then alternatively be formulated as follows: Given $\widetilde{\lambda}$, the expected queue length in the deviant submarket, the optimal k is such that the highest price or fee prevails, i.e., the optimal k solves the problem,

$$\max_{1 \le k \le a} \omega_k(\widetilde{\lambda}, \omega, \lambda). \tag{8}$$

If the price or fee in the deviant submarket is lower than the highest possible, then, as before, arbitrage is possible and buyers could receive more than their market utility. In this alternative formulation, we can think of the choice variable of sellers in the deviant submarket to be $\widetilde{\lambda}$ instead of $\widetilde{\omega}$ since choosing the optimal mechanism is isomorphic to choosing the optimal expected queue length that satisfies the constraint that when all buyers choose to visit submarkets optimally, none of them can get more than the market utility. Thus the expected payoff of deviant sellers can be expressed as a function of $\widetilde{\lambda}$,

$$\pi^d(\widetilde{\lambda}) = \pi_k(\omega_k(\widetilde{\lambda}, \omega, \lambda), \widetilde{\lambda}, \omega, \lambda)$$

where k solves the buyer's optimization problem. Below, we follow this approach since analytic expressions of $\omega_k(\widetilde{\lambda}, \omega, \lambda)$ can be obtained relatively easily.

2.3 The Social Planner's Problem

We first calculate total surplus and the marginal contribution to surplus by sellers and buyers. In the perfect complements case, surplus is generated if and only if a buyer has a offers. In the perfect substitutes case, surplus is generated if and only if a buyer has at least one offer. Note that because of constant returns to scale in the meeting technology, surplus per seller depends only on market tightness $\lambda = aB/S$ and can be written as

$$y(\lambda) = \begin{cases} \frac{\lambda}{a}\left(1 - h(\lambda)\right)^a = \frac{B}{S}\left(1 - h(\frac{aB}{S})\right)^a & \text{perfect complements} \\ \frac{\lambda}{a}\left(1 - h(\lambda)^a\right) = \frac{B}{S}\left(1 - h(\frac{aB}{S})^a\right) & \text{perfect substitutes} \end{cases} \tag{9}$$

since λ/a is the number of buyers per seller. Total surplus is then simply

$$V(B,S) = Sy\left(\frac{aB}{S}\right).$$

By direct computation, the marginal contribution to surplus of sellers, $\partial V(B,S)/\partial S$ or equivalently $y(\lambda) - \lambda y'(\lambda)$, is

$$V_s(\lambda) = \begin{cases} \left(1 - h(\lambda)\right)^{a-1} m(\lambda)\left(1 - \varepsilon_m(\lambda)\right) & \text{perfect complements} \\ h(\lambda)^{a-1} m(\lambda)\left(1 - \varepsilon_m(\lambda)\right) & \text{perfect substitutes} \end{cases} \tag{10}$$

With perfect complements, a seller contributes to surplus if the seller meets at least one buyer, which happens with probability $m(\lambda)$, and the chosen buyer has $a - 1$ other offers, which happens with probability $(1 - h(\lambda))^{a-1}$. In this scenario, a seller should obtain a share of the surplus that equals his or her marginal contribution to the matching process. The latter equals the elasticity of m with respect to the number of sellers, which equals $1 - \varepsilon_m(\lambda)$.

With perfect substitutes, a seller does not create surplus if the seller does not meet a buyer or if the seller meets a buyer but that buyer receives another offer. In that case, in the absence of the seller in question, the total amount of trade would be the same. Thus the seller contributes to surplus when he or she meets at least one buyer, which happens with probability $m(\lambda)$, and when that buyer has no other offers, which happens with probability $h(\lambda)^{a-1}$. Again when a seller contributes to surplus, the seller should receive a share equal to the elasticity of the meeting function, which is the seller's contribution to the meeting.

The marginal contribution to surplus by buyers, $\partial V(B,S)/\partial B$ or equivalently $ay'(\lambda)$ (since one buyer visits a sellers), is then

$$V_b(\lambda) = \begin{cases} \left(1 - h(\lambda)\right)^a\left(1 - a(1 - \varepsilon_m(\lambda))\right) & \text{perfect complements} \\ 1 - h(\lambda)^a - ah(\lambda)^{a-1}\left(1 - h(\lambda)\right)\left(1 - \varepsilon_m(\lambda)\right) & \text{perfect substitutes} \end{cases} \tag{11}$$

With perfect complements, a buyer contributes to surplus if he or she receives offers from a sellers, which happens with probability $(1 - h(\lambda))^a$. In this scenario, the buyer should receive the residual part of the surplus: $1 - a(1 - \varepsilon_m(\lambda))$, since each seller receives $1 - \varepsilon_m(\lambda)$. Note that $\lim_{\lambda \to 0} V_b(\lambda) = 1$ since at $\lambda = 0$, $h(0) = 0$ and $\varepsilon_m(0) = 1$. As λ increases, Lemma 2 below shows that $V_b(\lambda)$ strictly decreases until it reaches zero at a point $\overline{\lambda}^c$ where $\varepsilon_m(\overline{\lambda}^c) = 1 - 1/a$ (see equation (11) above). Since we assumed that $\varepsilon_m(\lambda)$ is strictly decreasing, when λ is larger than $\overline{\lambda}^c$, $V_b(\lambda)$ stays negative by equation (11) above. Furthermore, $\lim_{\lambda \to \infty} V_b(\lambda) = 0$ since $\lim_{\lambda \to \infty} h(\lambda) = 1$. With perfect complements, buyers impose a negative externality on each other: if two buyers visit the same seller, then the buyer who is not chosen by the seller will not be available for other sellers. Hence when the number of buyers is sufficiently large, their marginal contribution turns negative.

With perfect substitutes, when a buyer has strictly more than one offer, which happens with probability $1 - h(\lambda)^a - ah(\lambda)^{a-1}(1 - h(\lambda))$, then the buyer's contribution to surplus is 1; when the buyer has exactly one offer, which happens with probability $ah(\lambda)^{a-1}(1 - h(\lambda))$, his or her contribution to surplus is $\varepsilon_m(\lambda)$. As in the case of perfect complements, we have $\lim_{\lambda \to 0} V_b(\lambda) = 1$ and $\lim_{\lambda \to \infty} V_b(\lambda) = 0$. However, by Lemma 2 below, $V_b(\lambda)$ is always strictly decreasing. With perfect substitutes, a buyer who is not selected can still trade with another seller so the marginal contribution to surplus of buyers is always strictly positive in contrast to the case of perfect complements.

Lemma 2. *For perfect complements, $V_b(\lambda)$ is strictly positive for $\lambda < \overline{\lambda}^c$ and strictly negative for $\lambda > \overline{\lambda}^c$ where $\overline{\lambda}^c$ is uniquely defined by $\varepsilon_m(\overline{\lambda}^c) = 1 - 1/a$. Furthermore, $V_b(\lambda)$ is strictly decreasing in λ when $\lambda < \overline{\lambda}^c$.*

For perfect substitutes, $V_b(\lambda)$ is strictly decreasing in λ with $\lim_{\lambda \to 0} V_b(\lambda) = 1$ and $\lim_{\lambda \to \infty} V_b(\lambda) = 0$.

Proof. See Appendix A.1. □

Lemma 2 implies that for the case of perfect complements, $V_b(\lambda)$ is always strictly decreasing in λ when it is positive, which is equivalent to total surplus $y(\lambda)$ being strictly concave in λ when buyers' marginal contribution to surplus is positive. For the case of perfect substitutes, $V_b(\lambda) = ay'(\lambda)$ is always strictly decreasing in λ, which is equivalent to total surplus $y(\lambda)$ being strictly concave in λ.

The social planner's problem is to choose the number of buyers to maximize net output, i.e.,

$$\max_{B \geq 0} V(B, S) - KB$$

The socially optimal number of buyers is determined by the first-order condition $V_b(\lambda) = K$. By Lemma 2, this condition is also sufficient.

3 Sellers compete by entry fees

Suppose that the only mechanism, ω, that sellers can post is an entry fee t. In the case of perfect substitutes, an example of this is a simplified college admission problem. Identical colleges post application fees and no tuition. Students then apply to multiple colleges. This is a variation on Galenianos and Kircher (2009) with fee posting rather than wage posting. Colleges have a limited number of seats, which we set for simplicity to 1. Each college posts a fee t, which students have to pay whether they are admitted or not. Each college randomly selects a student from its applicants.

We first consider the buyer side. The expected payoff of a buyer who visits k deviant sellers with entry fee $\widetilde{t}$ and expected queue length $\widetilde{\lambda}$ and $n-k$ non-deviant sellers is

$$U_k(\widetilde{t},\widetilde{\lambda},t,\lambda)=\begin{cases}\left(1-h(\widetilde{\lambda})\right)^k(1-h(\lambda))^{a-k}-k\widetilde{t}-(a-k)t & \text{perfect complements}\\ 1-h(\widetilde{\lambda})^k h(\lambda)^{a-k}-k\widetilde{t}-(a-k)t & \text{perfect substitutes}\end{cases} \tag{12}$$

In the case of perfect complements, a buyer obtains a offers with probability $\left(1-h(\widetilde{\lambda})\right)^k(1-h(\lambda))^{a-k}$, and in the case of perfect substitutes, a buyer obtains at least one offer with probability $1-h(\widetilde{\lambda})^k h(\lambda)^{a-k}$.

The expected payoff of a deviant seller is

$$\pi_k(\widetilde{t},\widetilde{\lambda},t,\lambda)=\widetilde{\lambda}\widetilde{t}, \tag{13}$$

since each buyer has to pay $\widetilde{t}$ and the expected number of buyers for a deviant seller is $\widetilde{\lambda}$. Note that the above payoff does not depend on which other sellers the deviant seller's buyers visit.

In the next two subsections, we first solve for the competitive search Nash equilibrium and then for the competitive search market-maker equilibrium.

3.1 Competitive Search Nash Equilibrium

Consider a deviant seller in the competitive search Nash equilibrium. Equation (3) gives the relationship between the deviant seller's posted fee $\widetilde{t}$ and the expected queue length, $\widetilde{\lambda}$.

Consider first the case of perfect complements. Substituting equation (12) into (3) yields

$$(1-h(\lambda))^a-at=\left(1-h(\widetilde{\lambda})\right)(1-h(\lambda))^{a-1}-\widetilde{t}-(a-1)t,$$

where the left-hand side is $\overline{U}$. From this indifference condition, we can solve for $\widetilde{t}$ as a function of $\widetilde{\lambda}$ as follows:

$$\widetilde{t} = t + (h(\lambda) - h(\widetilde{\lambda}))(1 - h(\lambda))^{a-1}, \tag{14}$$

which implies that $\widetilde{t}$ is decreasing in $\widetilde{\lambda}$ since $h(\widetilde{\lambda})$ is an increasing function. The expected payoff of the deviant seller is

$$\pi^d(\widetilde{\lambda}) = \widetilde{\lambda}\widetilde{t} = \widetilde{\lambda}t + \widetilde{\lambda}\left[(1 - h(\widetilde{\lambda})) - (1 - h(\lambda))\right](1 - h(\lambda))^{a-1}, \tag{15}$$

where for analytical convenience, we have defined the expected seller payoff as a function of $\widetilde{\lambda}$ instead of $\widetilde{t}$. The above equation is strictly concave in $\widetilde{\lambda}$, which can be seen by noting that $\widetilde{\lambda}(1 - h(\widetilde{\lambda})) = m(\widetilde{\lambda})$ and the other terms are linear in $\widetilde{\lambda}$. Thus the first order condition is both necessary and sufficient.

In a symmetric pure-strategy equilibrium, the first order condition holds at $\widetilde{\lambda} = \lambda$. Hence, for the case of perfect complements, in equilibrium

$$t^* = \lambda h'(\lambda)(1 - h(\lambda))^{a-1}. \tag{16}$$

Substituting the above equation into (2) yields the market utility $\overline{U}$:

$$\overline{U} = (1 - h(\lambda))^a \left[1 - a\frac{\lambda h'(\lambda)}{1 - h(\lambda)}\right].$$

Note that by equation (1), we have $\lambda h'(\lambda)/(1 - h(\lambda)) = 1 - \varepsilon_m(\lambda)$. Therefore, for the case of perfect complements, we have

$$\overline{U} = V_b(\lambda),$$

where $V_b(\lambda)$ is the buyers' marginal contribution to surplus and is given by (11). Hence the competitive search Nash equilibrium is constrained efficient for the case of perfect complements.

We can now consider the case of perfect substitutes. Substituting equation (12) into (3) yields

$$1 - h(\lambda)^a - at = 1 - h(\widetilde{\lambda})h(\lambda)^{a-1} - \widetilde{t} - (a - 1)t.$$

Solving for $\widetilde{t}$ as a function of $\widetilde{\lambda}$ gives:

$$\widetilde{t} = t + (h(\lambda) - h(\widetilde{\lambda}))h(\lambda)^{a-1},$$

which again implies that $\widetilde{t}$ is decreasing in $\widetilde{\lambda}$ since $h(\widetilde{\lambda})$ is an increasing function. The expected payoff of the deviant seller is

$$\pi^d(\widetilde{\lambda}) = \widetilde{\lambda}\widetilde{t} = \widetilde{\lambda}t + \widetilde{\lambda}\left[(1 - h(\widetilde{\lambda})) - (1 - h(\lambda))\right] h(\lambda)^{a-1}. \tag{17}$$

As in the perfect complements case, the equation for the deviant seller's profit is strictly concave in $\widetilde{\lambda}$. Thus the first order condition is both necessary and sufficient.

In a symmetric pure-strategy equilibrium, the first order condition holds at $\widetilde{\lambda} = \lambda$. Hence, in equilibrium

$$t^* = \lambda h'(\lambda)h(\lambda)^{a-1}. \tag{18}$$

Substituting the above equation into (2) yields the market utility $\overline{U}$:

$$\overline{U} = 1 - h(\lambda)^a - a\lambda h'(\lambda)h(\lambda)^{a-1}. \tag{19}$$

Again by equation (1), we have that $\lambda h'(\lambda)/(1-h(\lambda)) = 1-\varepsilon_m(\lambda)$. Therefore, for both cases (perfect complements and perfect substitutes) we have

$$\overline{U} = V_b(\lambda),$$

where $V_b(\lambda)$ is the buyers' marginal contribution to surplus and is given by (11). Hence the competitive-search Nash equilibrium is constrained efficient for both cases.

3.2 Competitive search market-maker equilibrium

Now we consider the competitive search equilibrium when potential deviations are made by a market maker. We first solve the buyer optimality problem in (5). That is, if a buyer decides to visit the deviant submarket, what is the optimal number of visits or applications there? Following our discussion after equation (8), we can calculate $\omega_k(\widetilde{\lambda}, t, \lambda)$ from the indifference condition $U_0(\widetilde{t}, \widetilde{\lambda}, t, \lambda) = U_k(\widetilde{t}, \widetilde{\lambda}, t, \lambda)$ and then solve the optimization problem given by (8).

First consider the case of perfect complements. Since $U_k(\widetilde{t}, \widetilde{\lambda}, t, \lambda)$ is given by (12), from $U_0(\widetilde{t}, \widetilde{\lambda}, t, \lambda) = U_k(\widetilde{t}, \widetilde{\lambda}, t, \lambda)$ we can solve for $\widetilde{t}$:

$$\widetilde{t} = \omega_k(\widetilde{\lambda}, t, \lambda) = t - (1 - h(\lambda))^a \frac{1}{k}\left(1 - (\frac{1 - h(\widetilde{\lambda})}{1 - h(\lambda)})^k\right). \tag{20}$$

Next, consider the case of perfect substitutes. Again from $U_0(\widetilde{t}, \widetilde{\lambda}, t, \lambda) = U_k(\widetilde{t}, \widetilde{\lambda}, t, \lambda)$, we can solve for $\widetilde{t}$:

$$\widetilde{t} = \omega_k(\widetilde{\lambda}, t, \lambda) = t + h(\lambda)^a \frac{1}{k}\left(1 - \left(\frac{h(\widetilde{\lambda})}{h(\lambda)}\right)^k\right). \tag{21}$$

For both cases (perfect complements and perfect substitutes), the maximization problem in (8) can be easily solved with the help of the following technical result.

Lemma 3. *For $x > 0$ and $x \neq 1$, $\frac{1}{k}(1 - x^k)$ is strictly decreasing in k when $k > 0$.*

Proof. See Appendix A.2. □

The following lemma gives the solution to the buyer's problem and shows that it depends on whether the offers are complements or substitutes but not on the specific meeting technology.

Lemma 4. *Suppose sellers post only entry fees in the competitive search market-maker equilibrium.*

With perfect complements, buyers who visit the deviant submarket will pay all their visits to sellers in that submarket.

With perfect substitutes, buyers who visit the deviant submarket will visit only one seller in that submarket and make their other visits to sellers in the non-deviant submarket.

Proof. From Lemma 3, it follows that $\omega_k(\widetilde{\lambda}, t, \lambda)$ in (20) reaches its maximum at $k = a$. Similarly, $\omega_k(\widetilde{\lambda}, t, \lambda)$ in (21) reaches its maximum at $k = 1$. □

Strategic complements versus substitutes. To understand the intuition behind Lemma 4, consider the simplest case where $a = 2$. Both the original and a deviant submarket can be characterized by a state variable $\boldsymbol{x} = (h(\lambda), -t)$ (for the original submarket) and $\widetilde{\boldsymbol{x}} = (h(\widetilde{\lambda}), -\widetilde{t})$ (for the deviant submarket). Since a lower fee $\widetilde{t} < t$ in the deviant submarket implies a longer expected queue $\widetilde{\lambda} > \lambda$, we either have $\boldsymbol{x} > \widetilde{\boldsymbol{x}}$ or $\boldsymbol{x} < \widetilde{\boldsymbol{x}}$. Recall that $U_1(\widetilde{t}, \widetilde{\lambda}, t, \lambda)$, the expected payoff of paying exactly one visit to the deviant submarket, is given by equation (12) and can be rewritten as $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \mathcal{S}(h, \widetilde{h}) - t - \widetilde{t}$, where h and $\widetilde{h}$ is short-hand notation for $h(\lambda)$ and $h(\widetilde{\lambda})$, respectively, and $\mathcal{S}(h, \widetilde{h})$ is the expected surplus which is given by $(1-h)(1-\widetilde{h})$ for the case of perfect complements and by $1 - h\widetilde{h}$ for the case of perfect substitutes. The cross partial derivatives of $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$ are: $\mathcal{U}_{13}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \mathcal{S}_{12}(h, \widetilde{h})$, where $\mathcal{U}_{13}$ is the cross partial derivative with respect to h and $\widetilde{h}$ and $\mathcal{U}_{14} = \mathcal{U}_{23} = \mathcal{U}_{24} = 0$. Therefore, in the case of perfect complements, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic complements because $\mathcal{S}_{12}(h, \widetilde{h}) = 1 > 0$ while in the case of perfect substitutes, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic substitutes because $\mathcal{S}_{12}(h, \widetilde{h}) = -1 < 0$. In the former case, we have $\frac{1}{2}\mathcal{U}(\boldsymbol{x}, \boldsymbol{x}) + \frac{1}{2}\mathcal{U}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) > \mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$ and in the latter case, the reverse inequality holds.[7] Since fees are additively separable in the buyer payoff function $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$, only the surplus function $\mathcal{S}$ determines whether $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic complements or substitutes. When the goods are perfect complements, strategic complementarity in $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ implies that a buyer should either pay all visits to the non-deviant submarket or all visits to the deviant submarket (the exact number of buyers that the deviant submarket attracts follows from a no-arbitrage condition: $\mathcal{U}(\boldsymbol{x}, \boldsymbol{x}) = \mathcal{U}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{x}})$). When the goods are perfect substitutes, strategic substitutability implies that a buyer who visits the deviant submarket visits only one seller there and visits one seller in the non-deviant submarket. The fact that when goods are complements, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are also strategic complements is not obvious. In fact, we show below that when firms post only prices rather than fees, in the case of perfect complements $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic substitutes, while when the goods are perfect substitutes, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic complements.

Lemma 4 implies that with perfect substitutes, the Nash equilibrium and market-maker equilibrium coincide since even with a market maker, buyers who visit the deviant submarket pay exactly one visit to a deviant seller. This implies that the expected payoff of a deviant seller is the same in both types of equilibrium. However, with perfect complements a deviant seller's profit differs between the two cases. In the competitive search market-maker equilibrium, buyer optimality reduces to the condition $U_0(\widetilde{t}, \widetilde{\lambda}, t, \lambda) = U_a(\widetilde{t}, \widetilde{\lambda}, t, \lambda)$, which then determines $\widetilde{t}$ as a function of $\widetilde{\lambda}$:

$$\widetilde{t} = t + \frac{1}{a}\left((1 - h(\widetilde{\lambda}))^a - (1 - h(\lambda))^a\right). \tag{22}$$

In the proof of Proposition 1, which is given below, we show that if $\widetilde{\lambda} \neq \lambda$, the right-hand side of the above equation is strictly greater than the corresponding Nash case (equation (16)). Therefore, for any fixed $\widetilde{\lambda} \neq \lambda$, a deviant seller's expected profit is higher in the market-maker case and it is given by

$$\begin{aligned} \pi^d(\widetilde{\lambda}) &= \widetilde{\lambda}\widetilde{t} = \widetilde{\lambda}t + \frac{1}{a}\widetilde{\lambda}\left((1 - h(\widetilde{\lambda}))^a - (1 - h(\lambda))^a\right) = \frac{1}{a}\left[\widetilde{\lambda}(1 - h(\widetilde{\lambda}))^a - \widetilde{\lambda}\overline{U}\right] \\ &= y(\widetilde{\lambda}) - \frac{\widetilde{\lambda}}{a}\overline{U} \end{aligned} \tag{23}$$

[7]Consider a general payoff function $f(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^n$ are two variables that a decision maker has to choose. Following the literature, we call $\boldsymbol{x}$ and $\boldsymbol{y}$ strategic complements (resp. substitutes) in f if $\partial^2 f/(\partial x_i \partial y_j) \geq 0$ (resp. ≤ 0) for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Strategic complementarity is equivalent to the concept that $f(\boldsymbol{x}, \boldsymbol{y})$ has increasing differences in $(\boldsymbol{x}, \boldsymbol{y})$. That is, for any $\boldsymbol{x}' \geq \boldsymbol{x}$ and $\boldsymbol{y}' \geq \boldsymbol{y}$, we have $f(\boldsymbol{x}', \boldsymbol{y}') - f(\boldsymbol{x}, \boldsymbol{y}') \geq f(\boldsymbol{x}', \boldsymbol{y}) - f(\boldsymbol{x}, \boldsymbol{y})$. Note that in our setup, $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$ is always symmetric in $(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$: $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \mathcal{U}(\widetilde{\boldsymbol{x}}, \boldsymbol{x})$, so that increasing differences or equivalently strategic complementarity implies that $\mathcal{U}(\boldsymbol{x}, \boldsymbol{x}) + \mathcal{U}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) > 2\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$, where we have a strict inequality because each component of $\boldsymbol{x}$ is strictly greater than the corresponding component of $\widetilde{\boldsymbol{x}}$ and at least for one pair (i, j), the cross derivative $\partial^2 \mathcal{U}/(\partial x_i \partial \widetilde{x}_j)$ is strictly positive.

where $y(\widetilde{\lambda})$, the total surplus per seller, is defined in equation (9). The second line above shows that the expected payoff of a seller who joins the deviant submarket is the difference between total surplus and the "cost" of buyers where the price of one buyer is $\overline{U}$.

By Lemma 2 (the case of perfect complements), surplus per seller, $y(\widetilde{\lambda})$, is strictly concave in $\widetilde{\lambda}$ when $y'(\widetilde{\lambda})$ is positive, which implies that the first-order condition is both necessary and sufficient for the deviant seller's problem. In a symmetric pure-strategy equilibrium, the first-order condition holds at $\widetilde{\lambda} = \lambda$, Therefore, $ay'(\lambda) = \overline{U}$, or equivalently, the marginal contribution to surplus per buyer equals the market utility, and hence the decentralized equilibrium is constrained efficient. This implies that the equilibrium t is the same as the one given by equation (16). Hence, for the case of perfect complements, even though the two equilibrium concepts differ when we consider deviations, the equilibrium outcomes coincide. Proposition 1 summarizes this.

Proposition 1. *Assume that sellers post only fees. When the goods are perfect complements, the equilibrium outcomes of the two versions of competitive search equilibrium (Nash and market maker) coincide and both are constrained efficient. This occurs even though the expected payoff of a deviant seller is higher in the market-maker equilibrium for any posted fee* $\widetilde{t} \neq t$.

When the goods are perfect substitutes, then the outcomes, both on and off the equilibrium path, of the two versions of competitive search equilibrium (Nash and market maker) coincide. The (common) equilibrium is constrained efficient.

Proof. See Appendix A.3. □

The above logic for the efficiency result in the case of perfect complements can be generalized. Whenever buyers who visit the deviant submarket find it optimal to pay all their visits there, the expected payoff of deviant sellers can be written as $y(\widetilde{\lambda}) - \widetilde{\lambda}\overline{U}/a$, and the resulting equilibrium is constrained efficient. The logic is the same as in the familiar case in which a buyer can only visit one seller. A deviant seller's problem is analogous to one where the deviant seller can "buy" queues directly from a competitive market where the price for the expected queue length is $\overline{U}/a$. In the next section, in Lemma 7 and Proposition 3, we show that the same observation holds when sellers compete with prices and the goods are perfect substitutes. Thus whether a deviant submarket can attract all visits of buyers who decide to pay at least one visit there depends both on the surplus structure (perfect complements or perfect substitutes) and on the posted mechanism (fees or prices).

4 Price competition

4.1 Perfect complements

Suppose that sellers can post prices $(p_1, \ldots, p_a)$ but cannot charge fees and that the products that are offered for sale are perfect complements. Buyers purchase the product from a seller if and only if all of the buyer's a visits lead to offers. Hence, the prices $p_1, \ldots, p_{a-1}$ when the buyer has respectively $1, \ldots, a-1$ offers do not matter, only p_a matters. Without loss of generality, we assume that $p_1 = \cdots = p_a = p$, in which case the mechanism can be reinterpreted as *price posting*.

As before, first consider the buyer side. The expected payoff of a buyer who visits k deviant sellers with posted price $\widetilde{p}$ and expected queue length $\widetilde{\lambda}$ and $a-k$ non-deviant sellers with posted price p and expected queue length λ is

$$U_k(\widetilde{p}, \widetilde{\lambda}, p, \lambda) = \left(1 - h(\widetilde{\lambda})\right)^k (1 - h(\lambda))^{a-k} (1 - k\widetilde{p} - (a-k)p). \tag{24}$$

This expected payoff reflects the assumption that a buyer who receives offers from k deviant sellers and $a-k$ non-deviant sellers receives a payoff of $(1 - k\widetilde{p} - (a-k)p)$ times the corresponding probability of receiving those offers.

Next consider the seller side. Assume that all buyers that a deviant seller faces follow the same strategy: they all pay k visits to sellers in the deviant submarket $(\widetilde{p}, \widetilde{\lambda})$ and $a-k$ visits to the non-deviant submarket. Then the expected payoff of a deviant seller is

$$\pi_k(\widetilde{p}, \widetilde{\lambda}, p, \lambda) = m(\widetilde{\lambda}) \left(1 - h(\widetilde{\lambda})\right)^{k-1} (1 - h(\lambda))^{a-k} \widetilde{p}. \tag{25}$$

The deviant seller receives at least one visit with probability $m(\widetilde{\lambda})$ and makes a sale if and only if the chosen buyer receives $a-1$ other offers (perfect complements), which happens with probability $\left(1 - h(\widetilde{\lambda})\right)^{k-1} (1 - h(\lambda))^{a-k}$.

4.1.1 Competitive search Nash equilibrium

Consider a deviant seller who posts a price $\widetilde{p}$ and expects queue length $\widetilde{\lambda}$. In the Nash approach, only a single seller deviates, so buyers who visit the deviant seller must pay their other $a-1$ visits to the non-deviant submarket.

As before, from the buyer indifference condition, equation (3), we can solve for the relationship between $\widetilde{p}$ and $\widetilde{\lambda}$. Substituting equation (24) into equation (3) yields

$$\widetilde{p} = p + (1 - ap)\left(1 - \frac{1 - h(\lambda)}{1 - h(\widetilde{\lambda})}\right),$$

which implies that $\widetilde{p}$ is a decreasing function of $\widetilde{\lambda}$ (since $h(\widetilde{\lambda})$ is strictly increasing in $\widetilde{\lambda}$); i.e., a higher price leads to fewer buyer visits. Given the relationship between $\widetilde{\lambda}$ and $\widetilde{p}$, we can represent the expected payoff of the deviant seller as a function of $\widetilde{\lambda}$ only. That is, substituting the above equation into equation (25) with $k = 1$ yields

$$\pi^d(\widetilde{\lambda}) \equiv \pi_1(\widetilde{p}, \widetilde{\lambda}, p, \lambda) = (1 - h(\lambda))^{a-1}\left((1 - (a-1)p)\, m(\widetilde{\lambda}) - (1 - ap)\frac{m(\lambda)}{\lambda}\widetilde{\lambda}\right). \tag{26}$$

Since $m(\widetilde{\lambda})$ is strictly concave in $\widetilde{\lambda}$ and $1-(a-1)p > 1-ap \geq 0$, $\pi^d(\widetilde{\lambda})$ is also strictly concave in $\widetilde{\lambda}$, so the deviant seller's first-order condition is both necessary and sufficient. Taking the derivative with respect to $\widetilde{\lambda}$ yields

$$(1 - (a-1)p)\, m'(\widetilde{\lambda}) = (1 - ap)\frac{m(\lambda)}{\lambda}.$$

In a symmetric pure-strategy equilibrium, the first-order condition holds at $\widetilde{\lambda} = \lambda$. Thus in equilibrium we have

$$p^* = (1 - \varepsilon_m(\lambda))\,(1 - (a-1)p^*)\,, \tag{27}$$

where p^* denotes the equilibrium price. Note that we derived the equilibrium price p^* under the assumption that $a \geq 2$, but the above equation also gives the familiar result $p^* = 1-\varepsilon_m(\lambda)$ for the case of $a = 1$. When $a \geq 2$, a buyer purchases from a given seller if and only if all the buyer's other $a-1$ visits generate offers. From an individual buyer's and an individual seller's point of view, their match surplus is $1-(a-1)p^*$, given that the buyer needs to pay $(a-1)p^*$ to the other $a-1$ sellers; hence the equilibrium price equals this match surplus times $1-\varepsilon_m(\lambda)$. Simplifying the above equation yields

$$p^* = \frac{1 - \varepsilon_m(\lambda)}{a - (a-1)\varepsilon_m(\lambda)}. \tag{28}$$

Next, we compare the buyer's equilibrium payoff with his or her marginal contribution to surplus. By equation (11) (the case of perfect complements), the socially optimal price is

$$p^{SP} = 1 - \varepsilon_m(\lambda), \tag{29}$$

which is derived by setting $V_s(\lambda) = (1-h(\lambda))^{a-1} m(\lambda) p^{SP}$. From a social point of view, sellers should get their share of the surplus, which is $1-\varepsilon_m(\lambda)$. Therefore, the equilibrium price is lower than the socially optimal price, which implies that the expected payoff of buyers is strictly greater than their marginal contribution to surplus.

4.1.2 Competitive search market-maker equilibrium

In this case, a market maker can set up a deviant submarket. As before, we need to first solve the buyer portfolio problem. That is, if a buyer decides to visit the deviant submarket, what is the optimal number of sellers to approach in that submarket?

We can again first solve for $\widetilde{p}$ from the buyer's indifference condition, $U_0(\widetilde{p},\widetilde{\lambda},p,\lambda) = U_k(\widetilde{p},\widetilde{\lambda},p,\lambda)$. This gives

$$\widetilde{p} = \omega_k(\widetilde{\lambda},p,\lambda) = p + (1-ap)\frac{1}{k}\left(1-\left(\frac{1-h(\lambda)}{1-h(\widetilde{\lambda})}\right)^k\right). \tag{30}$$

With this expression, the buyers' maximization problem in (8) can be solved as before and the optimal $k=1$. It is never optimal to pay multiple visits to the deviant submarket, and we only have to consider the case where buyers pay exactly one visit there. The following lemma summarizes this case.

Lemma 5. *With perfect complements, when sellers compete with prices only in a competitive search market-maker equilibrium, buyers who visit the deviant submarket pay exactly one visit to the deviant submarket and their other visits to the non-deviant submarket.*

Proof. By Lemma 3, $\omega_k(\widetilde{\lambda},p,\lambda)$ in (30) reaches its maximum at $k=1$. □

For the case of perfect complements, the response from a buyer to a seller deviation when sellers post only prices is the opposite of the buyer response when sellers post only fees. When sellers post only prices, buyers who decide to visit the deviant submarket find it optimal to pay exactly one visit there even though the goods are perfect complements. To understand this, consider again the simplest case of $a=2$. Now define $\boldsymbol{x} = (h(\lambda), -p)$ and $\widetilde{\boldsymbol{x}} = (h(\widetilde{\lambda}), -\widetilde{p})$ where a lower price implies a longer queue so that we either have $\boldsymbol{x} > \widetilde{\boldsymbol{x}}$ or $\boldsymbol{x} < \widetilde{\boldsymbol{x}}$. By equation (24) with $k=1$ and $a=2$, we have that the buyer payoff of paying exactly one visit to the deviant submarket, $\mathcal{U}(\boldsymbol{x},\widetilde{\boldsymbol{x}}) = (1-h)(1-\widetilde{h})(1-p-\widetilde{p})$. In logs, this has a simple, additively separable structure: $\ln\mathcal{U}(\boldsymbol{x},\widetilde{\boldsymbol{x}}) = \ln(1-h) + \ln(1-\widetilde{h}) + \ln(1-p-\widetilde{p})$. The cross partial derivatives of $\ln\mathcal{U}$ are: $(\ln\mathcal{U})_{24}(\boldsymbol{x},\widetilde{\boldsymbol{x}}) = \partial^2 \ln\mathcal{U}/(\partial(-p)\partial(-\widetilde{p})) = -1/(1-p-\widetilde{p})^2 < 0$ and $(\ln\mathcal{U})_{13} = (\ln\mathcal{U})_{14} = (\ln\mathcal{U})_{23} = 0$. Therefore, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic substitutes in $\ln\mathcal{U}$. This implies that $\frac{1}{2}\ln\mathcal{U}(\boldsymbol{x},\boldsymbol{x}) + \frac{1}{2}\ln\mathcal{U}(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{x}}) < \ln\mathcal{U}(\boldsymbol{x},\widetilde{\boldsymbol{x}})$ when $\boldsymbol{x} \neq \widetilde{\boldsymbol{x}}$. To sum up, when the goods are perfect complements, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic substitutes. Consequently, a buyer who visits the deviant submarket will approach only one seller there and pay the other visit to the non-deviant submarket.

To better understand why buyer visits are strategic substitutes when sellers post prices only, imagine a deviant submarket with a lower price and a longer expected queue than the nondeviant submarket. A buyer who receives an offer from a seller in the deviant submarket can only realize this discount if he or she receives a second offer, and the probability of that is higher in the nondeviant submarket. Thus, the buyer sets $k = 1$. Now suppose instead that sellers post only fees. In this case, buyer visits are strategic complements. If a buyer visits a seller in a deviating submarket with a lower fee and longer expected queue, then the buyer pays this lower fee regardless of which other seller he or she visits and whether or not another offer is received. We have shown that it is more profitable for a buyer to either pay all visits to the non-deviant submarket or all visits to the deviant submarket (where the no-arbitrage condition guarantees that expected utility is the same as in the non-deviant submarket, i.e., fees drop sufficiently to compensate for the lower probability of two offers). Mixing between the low and high fee submarkets yields lower expected utility because of complementarity in the surplus function. That is, $k = 2$.

Lemma 5 implies that when considering deviations, the two concepts (market maker and Nash) coincide. The following Proposition summarizes our results.

Proposition 2. *Assume that sellers post prices $(p_1, \ldots, p_a)$ only. When the goods are perfect complements, then the outcomes, both on and off the equilibrium path, of the two versions of competitive search equilibrium (Nash and market maker) coincide. The (common) equilibrium is not constrained efficient, and the equilibrium payoff of buyers is strictly greater than their marginal contribution to surplus.*

Proof. See the above discussion. □

The inefficiency arises due to a externality that sellers impose on each other. With perfect complements, a buyer only buys if he or she receives an offer from all visited sellers. A seller who makes an offer to a buyer benefits from short queues at other sellers because then the buyer in question is more likely to receive other offers. Individual sellers however choose socially inefficient long queues and correspondingly low prices in order to secure trade (reduce the probability of zero arrivals). As a consequence, in equilibrium, sellers receive less than their social contribution to surplus. One may have expected that a market maker would be able to internalize this externality, but the market maker cannot force buyers to pay all their visits to a deviant submarket. It can offer sellers the opportunity to enter a deviant high-price, short-queue submarket, but sellers realize that buyers will still pay their other visits to non-deviant sellers, i.e., those with low prices and long queues.

4.2 Perfect substitutes

We now consider the case of price competition with perfect substitutes, which is an extension of the model presented in Albrecht et al. (2006). That model was set in the labor market and assumed an urn-ball meeting technology and only considered the competitive search Nash equilibrium.[8] Here we consider a product market with a general meeting technology, $m(\lambda)$, and show that the Nash equilibrium differs from the market-marker equilibrium, which is constrained efficient.

In the case of perfect substitutes, seller competition implies that a buyer with multiple offers receives the full value of the match ($p_2 = \cdots = p_a = 0$), and randomly chooses one of the competing sellers. To see this, suppose that in a candidate equilibrium $p_n > 0$ for some $n > 1$. Then a deviant seller in the Nash approach or a deviant market maker could choose $\widetilde{p}_n = p_n - \epsilon$ for some positive but sufficiently small ϵ and $\widetilde{p}_j = p_j$ for $j \neq n$. Buyer optimality then implies that $\widetilde{\lambda}$ and λ are arbitrarily close for sufficiently small ϵ. However, this would give a discrete increase in a deviant seller's winning probability and corresponding profit. Therefore, we must have $p_2 = \cdots = p_a = 0$. The following lemma states this result formally.

Lemma 6. *With perfect substitutes, when sellers compete with prices,* $p_2 = \cdots = p_a = 0$ *both in the competitive search Nash equilibrium and the competitive search market-maker equilibrium.*

Proof. See Appendix A.4. □

In equilibrium, a buyer receives no offers, and thus payoff zero, with probability $h(\lambda)^a$. The buyer receives exactly one offer, and thus payoff $1 - p_1$, with probability $ah(\lambda)^{a-1}(1 - h(\lambda))$, and the buyer receives multiple offers, and thus payoff 1, with the complementary probability $1 - h(\lambda)^a - ah(\lambda)^{a-1}(1 - h(\lambda))$. The market utility for buyers can be written as

$$\overline{U} = 1 - h(\lambda)^a - ah(\lambda)^{a-1}(1 - h(\lambda))p_1. \tag{31}$$

Similarly, the equilibrium payoff of a seller can be written as

$$\pi = m(\lambda)h^{a-1}p_1, \tag{32}$$

which equals the probability that the seller receives at least one visit times the probability that its selected buyer receives no other offers times p_1, the seller's payoff in that event.

We now consider the payoffs associated with deviations. As before, first consider a buyer who visits k deviant sellers who post $\widetilde{p}_1$ and have expected queue length $\widetilde{\lambda}$ and $a - k$ non-

[8] In Albrecht et al. (2006), workers send out multiple job applications, but they can only work for one firm, which then corresponds to the case of perfect substitutes here.

deviant sellers who post p_1 and have expected queue length λ, where $1 \le k \le a$. Then the buyer receives no offers, and thus payoff zero, with probability $h(\lambda)^{a-k}h(\widetilde{\lambda})^k$; one offer from a nondeviant together with no offer from a deviant, and thus payoff $1-p_1$, with probability $(a-k)h(\lambda)^{a-k-1}(1-h(\lambda))h(\widetilde{\lambda})^k$; one offer from a deviant together with no offer from nondeviants, and thus payoff $1-\widetilde{p}_1$, with probability $h(\lambda)^{a-k}kh(\widetilde{\lambda})^{k-1}(1-h(\widetilde{\lambda}))$; and multiple offers, and thus payoff 1 with the complementary probability. The buyer's expected payoff can then be written as

$$\begin{aligned} U_k(\widetilde{p}_1, \widetilde{\lambda}, p_1, \lambda) &= 1 - h(\lambda)^{a-k}h(\widetilde{\lambda})^k \\ &\quad - (a-k)h(\lambda)^{a-k-1}(1-h(\lambda))h(\widetilde{\lambda})^k p_1 - h(\lambda)^{a-k}kh(\widetilde{\lambda})^{k-1}(1-h(\widetilde{\lambda}))\widetilde{p}_1, \end{aligned} \tag{33}$$

where the first line on the right-hand side denotes the total surplus and the second line denotes the expected total payment.

Next, consider a deviant seller whose chosen buyer pays k visits to the deviant submarket and $a-k$ visits to the non-deviant submarket. The deviant seller's expected payoff is given by

$$\pi_k(\widetilde{p}_1, \widetilde{\lambda}, p_1, \lambda) = h(\lambda)^{a-k}h(\widetilde{\lambda})^{k-1}m(\widetilde{\lambda})\widetilde{p}_1, \tag{34}$$

where the deviant seller receives at least one visit with probability $m(\widetilde{\lambda})$ and the selected buyer's other visits fail with probability $h(\lambda)^{a-k}h(\widetilde{\lambda})^{k-1}$.

Before considering the two versions of competitive search equilibrium, note that given $p_2 = \cdots = p_a = 0$, the socially optimal p_1 can be derived from setting $V_b(\lambda) = 1 - h(\lambda)^a - ah(\lambda)^{a-1}(1-h(\lambda))p_1^{SP}$, where $V_b(\lambda)$ is given by equation (11) (the case of perfect substitutes), so

$$p_1^{SP} = 1 - \varepsilon_m(\lambda), \tag{35}$$

which is the same as the corresponding value for the case of perfect complements (see equation (29)).

4.2.1 Competitive search Nash equilibrium

First consider the Nash approach. As before, the buyer indifference condition (3) determines the relationship between the deviant seller's posted price $\widetilde{p}_1$ and the expected queue length

$\widetilde{\lambda}$. Substituting equation (33) into (3) yields $\widetilde{p}_1$ as a function of $h(\widetilde{\lambda})$ and hence of $\widetilde{\lambda}$:

$$\widetilde{p}_1 = p_1 + \frac{p_1(a(1-h)-1)+h}{h}\left(1-\frac{1-h}{1-\widetilde{h}}\right), \tag{36}$$

where, to simplify notation, we have replaced $h(\lambda)$ with h and $h(\widetilde{\lambda})$ with $\widetilde{h}$. Note that the coefficient in front of the large parenthesis on the right-hand side is linear in p_1, and it is strictly positive at $p_1 = 0$ and 1, which then implies that it is always strictly positive. Therefore, $\widetilde{p}_1$ is strictly decreasing in $\widetilde{h}$ and accordingly in $\widetilde{\lambda}$. That is, a higher price $\widetilde{p}_1$ leads to fewer buyer visits in expectation, i.e., a smaller $\widetilde{\lambda}$.

After substituting equation (36) into (34) (with $k = 1$), we can write the expected profit of the deviant seller as a function of $\widetilde{\lambda}$ only,

$$\pi^d(\widetilde{\lambda}) = \pi_1(\widetilde{p}_1, \widetilde{\lambda}, p_1, \lambda) = h^{a-2}(1-h)\left[\left((a-1)p_1 + \frac{h}{1-h}\right)m(\widetilde{\lambda}) - ((a-1)p_1 - (ap_1-1)h)\widetilde{\lambda}\right].$$

Since $m(\cdot)$ is strictly concave, $\pi^d(\widetilde{\lambda})$ is strictly concave in $\widetilde{\lambda}$, which implies that the first-order condition is both necessary and sufficient. Note that

$$\frac{d\pi^d(\widetilde{\lambda})}{d\widetilde{\lambda}}\Big|_{\widetilde{\lambda}=\lambda} = h^{a-1}(1-h)\left[p_1\left(1-(a-1)\frac{\lambda h'}{h}\right) - \frac{\lambda h'}{1-h}\right],$$

where we have used the fact that $m'(\lambda) = 1 - h(\lambda) - \lambda h'(\lambda)$. If the equilibrium $p_1 \in (0, 1)$ (interior solution), then the first-order condition requires that the above equation must be equal to zero. If the equilibrium $p_1 = 0$, then the above equation must be non-negative at $p_1 = 0$ (decreasing $\widetilde{\lambda}$ is not profitable). If the equilibrium $p_1 = 1$, then the above equation must be non-positive at $p_1 = 1$ (increasing $\widetilde{\lambda}$ is not profitable). Note that the right-hand side of the above equation is linear in p_1 and at $p_1 = 0$ it is strictly negative. If the derivative at $p_1 = 1$ it is strictly positive, then we have the interior solution. Otherwise, if at $p_1 = 1$ the derivative is negative, then we have a corner solution. To summarize, the equilibrium price p_1^* is given by

$$p_1^* = \begin{cases} \frac{\lambda h'}{1-h}/(1-(a-1)\frac{\lambda h'}{h}) & \text{if} \quad 1-(a-1)\frac{\lambda h'}{h} > \frac{\lambda h'}{1-h} \\ 1 & \text{otherwise.} \end{cases} \tag{37}$$

Note that for common meeting technologies such as the urn ball and geometric, for any λ, we have

$$1 - \frac{\lambda h'}{h} \leq \frac{\lambda h'}{1-h} \quad \Leftrightarrow \quad \frac{\lambda h'}{h} \geq 1-h. \tag{38}$$

Given inequality (38), the interior solution never occurs. That is, in equilibrium we always have $p_1^* = 1$,. This generalizes Albrecht et al. (2006) by allowing for a more general class of meeting technologies.

Finally, we contrast the competitive search Nash equilibrium price with the socially optimal p_1^{SP}. Since by equation (35), $p_1^{SP} = 1 - \varepsilon_m(\lambda) = \lambda h'/(1-h)$, we have

$$p_1^* > p_1^{SP}, \tag{39}$$

which holds, irrespective of whether or not equation (37) has an interior solution. Thus the equilibrium p_1^* is too high and a seller's equilibrium payoff is higher than their marginal contribution to surplus. This occurs because if all sellers charged the socially optimal price, each seller would have an incentive to deviate to a higher price. A higher price reduces the expected queue length less than it would in the case of a = 1, since when a ≥ 2, buyers have incentive to get multiple offers which would give them a price of zero. Buyers are therefore less deterred by a higher price. For the social planner, however, there is no value in multiple offers. That is, once a buyer has a first offer, a second offer adds nothing to total surplus; indeed, a second offer to one buyer makes a first offer to another buyer less likely. In short, the possibility of multiple offers means that the volume-margin tradeoff that the social planner faces is not the same as the one faced by buyers and sellers.

4.2.2 Competitive search market-maker equilibrium

We now consider the competitive search market-marker equilibrium. Since buyers can pay multiple visits to sellers in the deviant submarket, we first need to solve the buyer's problem in (8).

We follow the same procedure as before. First, from the buyer indifference condition, $U_0(\widetilde{p}_1, \widetilde{\lambda}, p_1, \lambda) = U_k(\widetilde{p}_1, \widetilde{\lambda}, p_1, \lambda)$, which is a linear equation in $\widetilde{p}_1$, we solve for $\widetilde{p}_1$ as a function of $\widetilde{\lambda}$:

$$\widetilde{p}_1 = \omega_k(\widetilde{\lambda}, p_1, \lambda) = \frac{1-h}{h}\left(\frac{\widetilde{h}}{1-\widetilde{h}}\right)p_1 - \frac{h+ap(1-h)}{h}\left(\frac{\widetilde{h}}{1-\widetilde{h}}\right)\frac{1}{k}\left(1-\left(\frac{h}{\widetilde{h}}\right)^k\right), \tag{40}$$

where we have again replaced $h(\lambda)$ by h and $h(\widetilde{\lambda})$ by $\widetilde{h}$. The above equation seems complicated but as a function of k it has a simple structure. Maximizing $\widetilde{p}_1$ with respect to k is equivalent to minimizing $\frac{1}{k}(1-(h/\widetilde{h})^k)$. Hence by Lemma 3, the optimal k for the buyer's maximization problem in (8) is $k = a$. The following lemma summarizes this result.

Lemma 7. *With perfect substitutes, when sellers compete with prices only in a competitive search market-maker equilibrium, buyers who visit the deviant submarket pay all their visits* *to the deviant submarket.*

Proof. By Lemma 3, $\omega_k(\widetilde{\lambda}, p_1, \lambda)$ in (40) reaches its maximum at $k = a$. □

As in the case of perfect complements, the choice of how many visits to pay to the deviant submarket differs depending on whether sellers post fees or prices for the case of perfect substitutes. Even though the goods are perfect substitutes, when sellers compete with prices, their offers are complementary to each other because buyers receive the full surplus with multiple offers. To see that the offers are indeed strategic complements, consider again the simplest case where $a = 2$. Define $\zeta = (1-h)(1-p_1)$, $\widetilde{\zeta} = (1-\widetilde{h})(1-\widetilde{p}_1)$, $\boldsymbol{x} = (h, \zeta)$, and $\widetilde{\boldsymbol{x}} = (\widetilde{h}, \widetilde{\zeta})$. By equation (33) with $k = 1$ and $a = 2$, we have that the buyer payoff of paying exactly one visit to the deviant submarket, $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = (1-h)(1-\widetilde{h}) + h\widetilde{\zeta} + \widetilde{h}\zeta$, where the first term on the right-hand side represents the expected value of two offers, the second term the expected value of one offer from the deviant submarket, and the last term the expected value of one offer from the non-deviant submarket. Note that ζ is the buyer's expected value of visiting the non-deviant submarket conditional on the visit to the deviant submarket failing to generate an offer. Since buyers always prefer short queues (low h) and high ζ, a higher $\widetilde{h} > h$ is always associated with a higher $\widetilde{\zeta} > \zeta$ and vice versa, This implies that we can either have $\boldsymbol{x} > \widetilde{\boldsymbol{x}}$ or $\boldsymbol{x} < \widetilde{\boldsymbol{x}}$. The cross partial derivatives of $\mathcal{U}$ are: $\mathcal{U}_{13} = \mathcal{U}_{14} = \mathcal{U}_{23} = 1$ and $\mathcal{U}_{24} = 0$. Therefore, even though the goods are perfect substitutes, $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are strategic complements in $\mathcal{U}$, and a buyer who visits the deviant submarket will pay both visits there. The main difference relative to the case in which firms compete with fees is that with price posting, $\mathcal{U}(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$ is no longer additive in surplus and payments but instead has interaction terms between matching probabilities and prices, which makes $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ strategic complements.

To better understand why buyer visits are strategic complements in this case, note that when the goods are perfect substitutes and firms post prices, buyers need only one successful offer, but they get the full surplus when they receive two offers. Imagine a deviant submarket with a higher price and a shorter expected queue than the nondeviant submarket. It is optimal for a buyer to pay a second visit to the deviant submarket with the shorter expected queue if he or she pays the first visit there, i.e., $k = 2$. It's as if a visit to the high-price, low-expected-queue submarket is an investment in which the buyer accepts the chance of paying a higher price if they get only one offer. Paying a second visit to this submarket raises the probability that the buyer will not have to pay the higher price but will rather get the whole surplus. This is why the visits are strategic complements. When firms post fees, the buyer is only interested in getting a single offer and buyer visits are thus strategic substitutes and $k = 1$.

Since buyers who visit the deviant submarket pay all their visits there, the expected profit of a deviant seller is

$$\pi^d(\widetilde{\lambda}) = y(\widetilde{\lambda}) - \frac{\widetilde{\lambda}}{a}\overline{U} \tag{41}$$

where total surplus per seller, $y(\widetilde{\lambda})$, is defined in equation (9), $\widetilde{\lambda}/a$ is the number of buyers per seller and $\overline{U}$ is the expected payoff that each buyer receives.[9] Thus the competitive search market-maker equilibrium is constrained efficient, and the equilibrium price $p_1^* = p_1^{SP} = 1 - \varepsilon_m(\lambda)$ (see equation (35)). The following proposition summarizes the results.

Proposition 3. *Assume that sellers post prices* $(p_1, \ldots, p_a)$ *only. When the goods are perfect substitutes, then the competitive search market-maker equilibrium is constrained efficient, whereas in the competitive search Nash equilibrium, the equilibrium payoff of buyers is strictly less than their marginal contribution to surplus.*

The key to this non-equivalence result has to do with the options available to buyers under the two interpretations of competitive search equilibrium. Consider a candidate competitive search equilibrium in which all sellers post p_1. To be an equilibrium, there must be no profitable deviation. Interpreting competitive search equilibrium as the limit of a sequence of Nash equilibria, this means that no single seller can profit by posting a price other than p_1. In this case, a buyer can pay at most one visit to a deviant seller. Using the market-maker interpretation, no profitable deviation means that a competing submarket, in which multiple sellers post a price other than p_1, cannot be profitably established. In this case, a buyer can choose any number of visits from $\{0, 1, ..., a\}$ to pay to deviant sellers. This expansion of buyer choice matters when there are interactions among a buyer's visits; that is, when the value of any one visit depends on the outcomes associated with his or her other visits.

The intuition for why the competitive search market-maker equilibrium is constrained efficient is that the market maker enables sellers to coordinate on p_1^{SP}. Each seller in a submarket where this price is posted knows that all other sellers in that submarket are posting the same price and that any buyer who visits this submarket is not visiting sellers in other submarkets. By joining the submarket with price p_1^{SP}, sellers are implicitly agreeing to cooperate with one another, i.e., to not raise p_1 above the social planner value. Relative to the Nash case, a seller in the submarket with price p_1^{SP} receives a lower price when his or her buyer has no other offer, but this is more than compensated for by a longer expected queue.

[9] Alternatively, we can use the buyers' indifference condition $U_a(\widetilde{p}_1, \widetilde{\lambda}, p_1, \lambda) = \overline{U}$ to solve for $\widetilde{p}_1$ as a function of $\widetilde{\lambda}$ and then substitute it into equation (34) with $k = a$, which then yields the expected payoff of a deviant seller:

$$\pi^d(\widetilde{\lambda}) = \widetilde{h}^{a-1} m(\widetilde{\lambda}) \frac{1 - \widetilde{h}^a - \overline{U}}{a\widetilde{h}^{a-1}(1 - \widetilde{h})} = y(\widetilde{\lambda}) - \frac{\widetilde{\lambda}}{a}\overline{U}.$$

5 Final remarks

In this paper, we have explored the foundations of competitive search equilibrium with simultaneous search; e.g., in the labor market, this occurs when workers apply to multiple vacancies and in the product market this occurs when buyers search at multiple sellers. It has previously been shown that when searchers engage in a single search, i.e., workers send one application or buyers visit only one seller, competitive search equilibrium is constrained efficient and the equilibrium allocation is the same according to a Nash concept of equilibrium or a market-maker concept of equilibrium. We show that these results do not always hold when agents make multiple searches. We find that if sellers post fees, but not prices, the competitive search equilibrium is constrained efficient and, as in the case of single searches, the Nash equilibrium and the market-maker equilibrium coincide, even though for the case of perfect complements, the two approaches yield different outcomes off the equilibrium path. If the sellers post prices and not fees, we find that if the goods in question are perfect complements, the Nash equilibrium and the market-maker equilibrium coincide but the equilibrium is not constrained efficient. If the goods are perfect substitutes, the Nash equilibrium allocation and the market-maker allocation are not the same, but the market-maker equilibrium is constrained efficient.

Appendix A Proofs

A.1 Proof of Lemma 2.

Recall that

$$V_b(\lambda) = \begin{cases} (1-h(\lambda))^a\,(1-a(1-\varepsilon_m(\lambda))) & \text{perfect complements} \\ 1-h(\lambda)^a - ah(\lambda)^{a-1}\,(1-h(\lambda))\,(1-\varepsilon_m(\lambda)) & \text{perfect substitutes} \end{cases}$$

We first consider the case of perfect complements. In this case, $V_b(\lambda)$ is positive if and only if $\lambda \leq \overline{\lambda}^c$, where $\varepsilon_m(\overline{\lambda}^c) = 1 - 1/a$. Recall that $\varepsilon_m(\lambda)$ is assumed to be strictly decreasing. Next, we have

$$V_b'(\lambda) = -a(1-h(\lambda))^{a-1}h'(\lambda)\,(1-a(1-\varepsilon_m(\lambda)) + (1-h(\lambda))^a a\varepsilon_m'(\lambda)$$

which is negative for $\lambda \leq \overline{\lambda}^c$.

Next, consider the case of perfect substitutes. To see that $V_b(\lambda)$ is positive, note that $1-h(\lambda)^a - ah(\lambda)^{a-1}\,(1-h(\lambda))\,(1-\varepsilon_m(\lambda)) > 1-h(\lambda)^a - ah(\lambda)^{a-1}\,(1-h(\lambda)) > 0$, where the first inequality holds because $\varepsilon_m(\lambda)$ is between 0 and 1, and the second inequality holds because its left-hand side represents the probability that a buyer receives at least two offers, which is strictly positive for $a \geq 2$. Furthermore, we have

$$V_b'(\lambda) = -ah(\lambda)^{a-1}\left(h'(\lambda)\varepsilon_m(\lambda) + (a-1)\left(\frac{1-h(\lambda)}{h(\lambda)}\right)h'(\lambda)(1-\varepsilon_m(\lambda)) - (1-h(\lambda))\varepsilon_m'(\lambda)\right)$$

which is strictly negative. □

A.2 Proof of Lemma 3.

The derivative of $\frac{1}{k}(1-x^k)$ with respect to k is:

$$\frac{\partial}{\partial k}\left(\frac{1}{k}(1-x^k)\right) = -\frac{1}{k^2}\left(1 - x^k + kx^k \ln x\right)$$

Define $y = k\ln x$ (equivalently $x^k = e^y$), then $1 - x^k + kx^k \ln x = 1 - e^y + ye^y$. Note that $\frac{d}{dy}(1-e^y+ye^y) = ye^y$, which is strictly positive when $y > 0$ and strictly negative when $y < 0$. Thus at $y = 0$, $1-e^y+ye^y$ reaches its minimum value, which is zero. When $x > 0$ and $x \neq 1$, $y \neq 0$, and consequently $\frac{1}{k}(1-x^k)$ is strictly decreasing in k. □

A.3 Proof of Proposition 1.

For the case of perfect complements, we showed in the text that the Nash and market-maker equilibria have the same outcomes which are constrained efficient. What remains to be shown is that the expected payoff of a deviant seller is strictly higher in the market-maker case than in the Nash case for any $\widetilde{t} \neq t$.

In the Nash case, the relationship between $\widetilde{t}$ and $\widetilde{\lambda}$ is given by equation (14), and in the market-maker case, it is given by equation (22). Denote the former relation by $\widetilde{t} = f_N(\widetilde{\lambda})$ and the latter by $\widetilde{t} = f_{MM}(\widetilde{\lambda})$; both are strictly decreasing functions. Note that

$$f_{MM}(\widetilde{\lambda}) - f_N(\widetilde{\lambda}) = \left[t + \frac{1}{a}\left((1-h(\widetilde{\lambda}))^a - (1-h(\lambda))^a\right)\right] - \left[t + (h(\lambda) - h(\widetilde{\lambda}))(1-h(\lambda))^{a-1}\right]$$

To simplify the notation, we write $\widetilde{h}$ for $h(\widetilde{\lambda})$ and h for $h(\lambda)$. The derivative of the above expression with respect to $\widetilde{h}$ is then given by

$$(1-h)^{a-1} - (1-\widetilde{h})^{a-1}$$

which is strictly negative when $\widetilde{h} < h$ and strictly positive when $\widetilde{h} > h$. Hence the difference $f_{MM}(\widetilde{\lambda}) - f_N(\widetilde{\lambda})$ reaches its minimum zero at $\widetilde{h} = h$ or equivalently $\widetilde{\lambda} = \lambda$. When $\widetilde{\lambda} \neq \lambda$ ($\widetilde{h} \neq h$), it is strictly positive.

Finally, the results for the case of perfect substitutes are shown in the text. □

A.4 Proof of Lemma 6.

Consider first the market-maker case. Suppose that in equilibrium $p_n > 0$ for some $n \in \{2, \ldots, a\}$. Consider a market maker who creates a deviant submarket with $\widetilde{p}_n = p_n - \epsilon$ for some positive but sufficiently small ϵ and $\widetilde{p}_j = p_j$ for $j \neq n$. Suppose that the expected queue length in the deviant submarket is $\widetilde{\lambda}$. Note that if the prices are arbitrarily close, the queues will be arbitrarily close ($\widetilde{\lambda} \to \lambda$ as $\epsilon \to 0$).

Let $\mathcal{P}(j_1, j_2, \lambda)$ represent the probability that a buyer who pays j_1 visits to a submarket with expected queue length λ obtains exactly j_2 offers. Then for $0 \leq j_2 \leq j_1$, we have

$$\mathcal{P}(j_1, j_2, \lambda) = \binom{j_1}{j_2} (1 - h(\lambda))^{j_2}\, h(\lambda)^{j_1 - j_2}. \tag{42}$$

Given ϵ, suppose that all buyers who visit the deviant submarket find it optimal to pay k visits to the deviant submarket and $a - k$ visits to the nondeviant submarket, then the expected payoff of a deviant seller is

$$\widetilde{\pi} = m(\widetilde{\lambda})\Bigg[\Bigg(\sum_{i=1}^{\min\{k,n\}} \frac{p_n - \epsilon}{i} \mathcal{P}(k, i-1, \widetilde{\lambda})\mathcal{P}(a-k, n-i, \lambda)\Bigg) + \sum_{1 \leq j \leq a,\, j \neq n} \frac{p_j}{j} \Bigg(\sum_{i=1}^{\min\{k,j\}} \mathcal{P}(k, i-1, \widetilde{\lambda})\mathcal{P}(a-k, j-i, \lambda)\Bigg)\Bigg]$$

where $\mathcal{P}(\cdot, \cdot, \cdot)$ is defined by equation (42). The first line above represents the scenario in which the deviant seller meets at least one buyer (with probability $m(\widetilde{\lambda})$) and the selected buyer has n offers in total. Among the n offers, i offers come from deviant sellers and $n - i$ offers come from nondeviant sellers. This occurs with probability $\mathcal{P}(k, i-1, \widetilde{\lambda})\mathcal{P}(a-k, n-i, \lambda)$, where $i \leq \min\{k, n\}$. In this case, the buyer randomizes among the i offers from the deviant submarket because the price $\widetilde{p}_n$ is strictly less than p_n. The second line represents the scenario where the deviant seller's chosen buyer has j offers in total ($j \neq n$). Among the j offers, i offers come from deviant sellers and $j - i$ offers come from the nondeviant sellers. This occurs with probability $\mathcal{P}(k, i-1, \widetilde{\lambda})\mathcal{P}(a-k, j-i, \lambda)$, where $i \leq \min\{j, k\}$. In this case, the buyer randomizes among all j offers because $\widetilde{p}_j = p_j$.

A seller's expected payoff in the candidate equilibrium can be obtained by setting $\widetilde{\lambda} = \lambda$ and replacing the term $(p_n - \epsilon)/i$ with p_n/n. Therefore, when $\epsilon \to 0$, the difference between the expected profits of a deviant seller and a nondeviant seller is

$$\lim_{\epsilon \searrow 0} (\widetilde{\pi} - \pi) = m(\lambda) \left(\sum_{i=1}^{\min\{k,n\}} \left(\frac{p_n}{i} - \frac{p_n}{n} \right) \mathcal{P}(k, i-1, \widetilde{\lambda}) \mathcal{P}(a-k, n-i, \lambda) \right).$$

Thus, there exist profitable deviations for sufficiently small ϵ (since all terms are positive), and this does not depend on the visiting strategies of buyers who send at least one visit to the deviant submarket.

Note that the above proof with $k = 1$ applies to the Nash case so we do not need a separate proof. □

References

Albrecht, J., Cai, X., Gautier, P., and Vroman, S. (2020). Multiple applications, competing mechanisms, and market power. *Journal of Economic Theory*, 190:105121.

Albrecht, J. W., Gautier, P. A., and Vroman, S. B. (2006). Equilibrium directed search with multiple applications. *Review of Economic Studies*, 73:869–891.

Burdett, K., Shi, S., and Wright, R. (2001). Pricing and matching with frictions. *Journal of Political Economy*, 109:1060–1085.

Eeckhout, J. and Kircher, P. (2010). Sorting vs screening - search frictions and competing mechanisms. *Journal of Economic Theory*, 145:1354–1385.

Galenianos, M. and Kircher, P. (2009). Directed search with multiple job applications. *Journal of Economic Theory*, 114:445–471.

Galenianos, M. and Kircher, P. (2012). On the game-theoretic foundations of competitive search equilibrium. *International economic review*, 53(1):1–21.

Kircher, P. (2009). Efficiency of simultaneous search. *Journal of Political Economy*, 117:861–913.

Moen, E. R. (1997). Competitive search equilibrium. *Journal of Political Economy*, 105:385–411.

Mortensen, D. T. and Wright, R. (2002). Competitive pricing and efficiency in search equilibrium. *International Economic Review*, 43(1):1–20.

Peters, M. (2000). Limits of exact equilibria for capacity constrained sellers with costly search. *Journal of Economic Theory*, 95:139–168.

Wright, R., Kircher, P., Julien, B., and Guerrieri, V. (2021). Directed search and competitive search equilibrium: A guided tour. *Journal of Economic Literature*, 59(1):90–148.