PHBS WORKING PAPER SERIES

Meetings and Mechanisms

Xiaoming Cai
Peking University

Pieter Gautier
VU University Amsterdam
Tinbergen Institute

Ronald Wolthoff
University of Toronto

July 2021

Working Paper 20210704

Abstract

There exists a lot of variation in terms of how buyers and sellers meet, both across sectors and over time as new technologies arrive. Rationed sellers care about how many buyers visit because this affects the expected highest valuation. In this paper we construct a new method to characterize the meeting technology and then ask, in the context of a directed-search model with competing mechanisms, how it affects market segmentation and the equilibrium selling mechanisms. Under mild conditions, high-valuation buyers are all located in the same segment. Then, we show under what conditions, low valuation buyers are in: (i) the same segment, (ii) a different segment and (iii) a mixture of (i) and (ii). The decentralized equilibrium is always efficient when sellers can post auctions with reverse prices or entry fees.

Keywords: Search frictions, directed search, matching function, meeting technology, competing mechanisms, competing auctions
JEL Classification: C78, D44, D83

Peking University HSBC Business School
University Town, Nanshan District
Shenzhen 518055, China

Meetings and Mechanisms*

Xiaoming Cai† Pieter Gautier‡ Ronald Wolthoff§

June 9, 2021

Abstract

There exists a lot of variation in terms of how buyers and sellers meet, both across sectors and over time as new technologies arrive. Rationed sellers care about how many buyers visit because this affects the expected highest valuation. In this paper we construct a new method to characterize the meeting technology and then ask, in the context of a directed-search model with competing mechanisms, how it affects market segmentation and the equilibrium selling mechanisms. Under mild conditions, high-valuation buyers are all located in the same segment. Then, we show under what conditions, low valuation buyers are in: (i) the same segment, (ii) a different segment and (iii) a mixture of (i) and (ii). The decentralized equilibrium is always efficient when sellers can post auctions with reserve prices or entry fees.

JEL classification: C78, D44, D83.
Keywords: search frictions, directed search, matching function, meeting technology, competing mechanisms, competing auctions.

*We thank Philipp Kircher, Michael Peters, Gábor Virág, and various seminar and conference participants for valuable comments. Ronald Wolthoff gratefully acknowledges financial support from the Connaught Fund and the Social Sciences and Humanities Research Council.
†Peking University HSBC Business School. E-mail: xmingcai@gmail.com.
‡VU University Amsterdam and Tinbergen Institute. E-mail: p.a.gautier@vu.nl.
§University of Toronto. E-mail: ronald.p.wolthoff@gmail.com.

1 Introduction

Real-life markets display a large degree of heterogeneity in the way in which economic agents meet and trade with each other: for example, in traditional bazaars, meetings between buyers and sellers tend to be bilateral; in real estate markets, multiple buyers may bid on the same house; and in labor markets, a typical vacancy receives a large number of applications but only interviews a subset.[1] Similarly, there is variation over time as the internet has made it easier for agents to meet multiple potential trading partners simultaneously; prominent examples of platforms utilizing this feature include eBay in the product market, Match.com in the dating market, CareerBuilder in the labor market, and Google AdWords in the market for online advertising. In this paper, we construct a formal framework in which the selling mechanism and market segmentation respond to changes in the meeting technology. Although our main contribution is theoretical, our model can help us understand those observations.

Economic theory has been mostly silent on the question how agents in these markets get to meet each other and how this meeting process affects equilibrium outcomes. This silence is most apparent in work that sidesteps a detailed description of the meeting process altogether by assuming a Walrasian equilibrium. Perhaps more surprisingly, the search literature—which aims to analyze trade in the absence of a Walrasian auctioneer—does not provide much more guidance: without much motivation, the vast majority of papers in this literature simply assumes one of two specific meeting technologies: either meetings between agents are one-to-one (bilateral meetings) or they are n-to-1, where n follows a Poisson distribution (urn-ball meetings).[2]

This approach seems restrictive for a number of reasons. First, neither bilateral meetings nor urn-ball meetings are necessarily an adequate description of real-life markets; in many cases, e.g. in the labor market example above, it appears necessary to consider alternatives. Second, assuming a particular meeting technology inevitably affects aggregate outcomes, i.e. crowding out of high-type agents by low-type agents is a larger concern when meetings are bilateral than when they are many-on-one.

We aim to make progress by presenting a unified framework that allows for a wide class of meeting technologies. We do so in an environment in which a continuum of buyers with ex ante heterogeneous private valuations and a continuum of identical sellers try to trade.[3]

[1] See Geertz (1978) for a characterization of the market interaction at a bazaar, Han and Strange (2014) for empirical evidence on bidding wars in real estate markets, and Wolthoff (2018) and Davis and de la Parra (2017) for evidence on applications and interviews in the labor market.

[2] Bilateral meetings can be found in e.g. Albrecht and Jovanovic (1986), Moen (1997), Guerrieri et al. (2010), and Menzio and Shi (2011). Urn-ball meetings are used in e.g. Peters (1997), Burdett et al. (2001), Shimer (2005), Albrecht et al. (2014) and Auster and Gottardi (2017). In addition, some papers in the mechanism design literature explore urn-ball meetings in a finite market, making n binomial rather than Poisson, by allowing for entry of buyers into a monopolistic auction Levin and Smith (1994).

[3] The fact that buyers know their valuation before visiting a seller distinguishes our work from Lester et al.

The class of technologies that we consider allows for various types of meeting externalities. For example, when a buyer tries to meet a seller, this action may reduce the likelihood that the seller can meet another buyer because of congestion externalities (e.g. websites become inaccessible when many buyers try to purchase tickets for a popular show). While meeting externalities are often negative in real-life, our model also allows for positive meeting externalities. The well-known bilateral and urn-ball meeting technologies as well as other meeting technologies in the literature are all special cases of our general meeting technology. This allows us to not only clarify existing results but to also analyze which of them carry over to our more general setting where sellers can meet multiple buyers but where there also is some rationing.

For some results, this generalization matters. For example, the finding that reserve prices are driven to the seller's valuation in an environment with competing auctions—see e.g. Albrecht et al. (2012)—only holds for some meeting technologies but not for others. In addition, we show that some meeting technologies give rise to partial separation rather than complete pooling or complete separation, which have been the focus of the literature so far (see e.g. Eeckhout and Kircher, 2010b; Cai et al., 2017).

The equilibrium mechanism that we identify includes both auctions without fees or explicit reserve prices (e.g. when meetings are urn-ball) and posted prices (when meetings are bilateral) as special cases.[4] Varying the degree of search frictions in our model changes the optimal mechanism. This interaction contrasts with much of the search literature (with the exception of some of the above papers), which assumes that the trading mechanism (e.g. bilateral bargaining) is independent of the frictions. However, changes in mechanisms due to changes in the meeting process are frequently observed in real-life. For example, as soon as eBay provided a platform for sellers and buyers to meet, auctions quickly gained popularity for the sale of e.g. second-hand products.[5] Einav et al. (2017) argue however that in recent years the popularity of auctions on eBay has declined relative to posted prices, which they explain by an increase in the *hassle cost* associated with purchasing in an auction (see Backus et al. (2015) for a particular example of such a cost). However, their study restricts attention to cases in which a seller sells multiple units of the same product (mostly retail items). They acknowledge that auctions remain the trading mechanism of choice for most sellers with a single unit, which is the case that we consider here. Note further that various other platforms, e.g. Catawiki or liveauctioneers continue to exclusively use auctions. In order to highlight the role of meeting technologies, we therefore abstract from hassle costs here.

(2015). See below for a more detailed comparison.

[4]When meetings are bilateral, buyers either bid the reserve price or pay the entrance fee and bid 0; both are equivalent to a posted price.

[5]Lucking-Reiley (2000) presents various statistics regarding the growing popularity of online auctions in the late 1990s.

Our model also helps to understand what happened in the market for freelance services, where new platforms like Upwork (previously oDesk) or Freelancer enable employers from high-income countries to outsource tasks to contractors from mainly low-income countries (see for a detailed description Agrawal et al., 2015).[6] These online platforms facilitate many-to-one meetings (also for small firms), creating scope for wage mechanisms other than bilateral bargaining. In particular, contractors apply to posted jobs by submitting a cover letter and a bid indicating the compensation that they demand for the job, after which procurers select one of the applicants.[7] These examples nicely illustrate how a new technology can affect the meeting process and how the market responds by adjusting the price or wage mechanism accordingly. Although our framework abstracts from many details of real-life markets, it helps to understand this novel fact.

Most related to our work is Eeckhout and Kircher (2010b) who showed that the meeting technology matters for posted mechanisms and market segmentation. The pioneering work of McAfee (1993), Peters (1997) and Peters and Severinov (1997) on competing auctions has focused on urn-ball meetings, with Albrecht et al. (2014) being a recent example. Relative to Eeckhout and Kircher (2010b), we fully characterize the equilibrium. That is, we show under what conditions different forms of market segmentation arise. Lester et al. (2015) provide a full characterization of the equilibrium, but in a simpler environment in which all buyers are ex ante identical and learn their type only after meeting a seller, which results in all agents participating in the same (sub)market in equilibrium. In contrast, models with ex ante heterogeneity, as we consider here, yield very different equilibrium outcomes: buyers and sellers must determine with whom they are willing to interact and multiple submarkets may arise.

Our paper makes three main contributions. First, we go further than the existing literature and characterize equilibrium for a wide class of meeting technologies, including cases where a seller can meet multiple but not all buyers, so low-type buyers may crowd out high-type buyers. For those cases, we establish that partial market segmentation may arise: all high-type buyers and a subset of the low-type buyers form a submarket, while the remaining low-type buyers form a separate submarket. Whether this outcome is obtained, or rather complete pooling or complete separation, or an equilibrium in which low-type buyers stay out

[6] Although still relatively new, these platforms already have a substantial impact on this market. The number of hours worked at Upwork increased by 55% between 2011 and 2012, with the 2012 total wage bill being more than 360 million dollar. A 2014 New York Times article states: "It's also helping to raise the standard of living for workers in developing countries. The rise of these marketplaces will increase global productivity by encouraging better matching between employers and employees." (Korkki, 2014).

[7] A more exotic example from the dating market is the following case where Amir Pleasants, a 21-year woman from New Jersey invited 150 men on a Tinder date to meet in Union square where she organized a pop-up dating competition where first all guys who were shorter than 5 foot 10 were eliminated and after a number of other rounds she ultimately selected a single winner. See https://www.nytimes.com/2018/08/20/style/tinder-dating-scam-union-square.html

of the market, depends both on the meeting technology and the dispersion in buyer types, and we provide a precise characterization. Motivated by the above real-world life examples, we then show how the meeting technology affects the optimal selling mechanism and market segmentation. For example, we discuss how the equilibrium changes when sellers can screen more buyers or when they can better distinguish the low-valuation from the high-valuation buyers. We also consider how a change in the spread of buyer valuations affects equilibrium. This exercise could be interpreted as an increase in inequality or globalization which leads to more dispersion in the population of buyers.

Cai et al. (2017) apply the tools that are developed in this paper (which we discuss in more detail below) to derive conditions on the meeting technology for which the equilibrium features either perfect separation or perfect pooling of different types of buyers; they find that perfect separation occurs if and only if meetings are bilateral (i.e. sellers can meet at most one buyer), and perfect pooling arises if and only if the meeting technology is jointly concave.[8] Cai et al. (2017) do not discuss more realistic meeting technologies where sellers can meet multiple but not all buyers, which are the main focus of our work here. We show that for these technologies partial segmenetation arises, of which the results on perfect separation and pooling discussed in Cai et al. (2017) are two special cases.

Second, we make a methodological contribution. In particular, we introduce an alternative representation of meeting technologies which keeps the analysis tractable. This representation is the probability ϕ that a seller meets at least one buyer from a given subset; usually, the relevant subset consists of buyers with a valuation above a certain threshold. This probability depends on two arguments: the total queue length λ that the seller faces as well as the queue of buyers μ belonging to the subset. We show that using ϕ instead of the more standard representation of meeting technologies offers a few important advantages. First, the partial derivatives of ϕ have natural interpretations corresponding to key variables such as a buyer's winning probability and the degree of meeting externalities. Second, expected surplus is linear in ϕ, which makes it straightforward to relate the objective of a planner to properties of ϕ.[9] Finally, the use of ϕ guarantees that the expression for a seller's payoff retains a similar structure as in the seminal work by Myerson (1981), i.e. as the integral of buyers' virtual valuation with respect to the distribution of highest valuations, with the difference that this distribution now also depends on how likely each buyer is to meet a seller which in turn depends on the meeting technology. In other words, the introduction of ϕ adds a lot of generality to the competing mechanism literature at relatively low cost.

Finally, our efficiency result contributes to the literature on directed search. In particular,

[8] They also relate those conditions to other properties of meeting technologies that have been derived in the literature, like invariance Lester et al. (2015) and non-rivalry Eeckhout and Kircher (2010b).

[9] Cai et al. (2017) exploit this feature in their work.

it extends the result by Albrecht et al. (2014) that all agents earn their marginal contribution to surplus in the special case in which meetings are urn-ball and sellers post regular auctions.[10] In that environment, there are no meeting externalities, so a buyer contributes to surplus only if he has the highest valuation among all buyers meeting a seller. The general case is more complicated because now a buyer can also impose positive or negative meeting externalities on meetings between the seller and other buyers, which should be reflected in the equilibrium payoffs. We show that an appropriate reserve price or meeting fee/subsidy is both profit maximizing and socially efficient.[11] As a result, all agents continue to receive their marginal contribution to surplus and efficiency survives.

After describing the environment and the alternative representation of the meeting technology in detail in section 2, we start our analysis in section 3 by solving the problem of a social planner. Section 4 shows how the planner's solution can be decentralized. It provides a characterization of the equilibrium, and it shows that under mild restrictions on the meeting technology there exists a unique queue for each fee/subsidy or reserve price (in combination with an auction) that a seller posts. In section 5, we show how our main results can be generalized to N buyer types.

2 Model

2.1 Environment

Agents and Preferences. A static economy is populated by risk-neutral buyers and sellers. Each seller possesses a single unit of an indivisible good, for which each buyer has unit demand. All sellers have the same valuation for their good, which we normalize to zero. Buyers are heterogeneous in their valuation x, which takes one of two different values, satisfying $0 < x_1 < x_2$. We will generally normalize x_1 to 1, turning x_2 into a measure of the dispersion in valuations. The measure of sellers is 1; the measure of buyers with value x_k is B_k for $k \in \{1, 2\}$. Buyers' valuations are private information and the market is anonymous in the sense that buyers and sellers cannot condition their strategies on the identities of their counterparties.

Mechanisms. In the first stage, each seller posts and commits to a direct anonymous mechanism to attract buyers. The mechanism specifies, for each buyer i, a probability of trade and an expected payment as a function of: (i) the total number n of buyers that

[10] Although we assume a fixed number of sellers to simplify exposition, our results carry over to an environment with free entry of sellers, as in Albrecht et al. (2014), in a straightforward manner.

[11] The reserve price or fee can vary across sellers in equilibrium. This is a key difference with Lester et al. (2015), where the fee is the same for all sellers as it only depends on exogenous parameters.

successfully meet with the seller; (ii) the valuation v_i that buyer i reports; and (iii) the valuations v_{-i} reported by the $n-1$ other buyers.[12]

Search. After observing all mechanisms, each buyer chooses the one at which he wishes to attempt to match. To capture the idea that coordination is not feasible in a large market, we follow the literature (see e.g. Montgomery, 1991; Burdett et al., 2001; Shimer, 2005) and restrict buyers to symmetric strategies. We refer to all buyers and sellers choosing a particular mechanism as a *submarket*.

Meeting Technology. Consider a submarket with a measure b of buyers and a measure s of sellers. The meetings within the submarket are frictional and governed by a *meeting technology*, which we model analogous to Eeckhout and Kircher (2010b). The meeting technology is anonymous; it treats all buyers (sellers) in a symmetric way, i.e., independent of their identity. A buyer can meet at most one seller, while a seller may meet multiple buyers. Define $\lambda = b/s$ as the *queue length* in this submarket.[13] The probability of a seller meeting n buyers, $n = 0, 1, 2, \ldots$, is given by $P_n(\lambda)$, which is assumed to be continuously differentiable.[14] Because each buyer can meet at most one seller, $\sum_{n=1}^{\infty} nP_n(\lambda) \leq \lambda$. By an accounting identity, the probability for a buyer to be part of an n-to-1 meeting is $Q_n(\lambda) \equiv nP_n(\lambda)/\lambda$ with $n \geq 1$. Finally, the probability that a buyer fails to meet any seller is $Q_0(\lambda) \equiv 1 - \sum_{n=1}^{\infty} Q_n(\lambda)$.[15]

Strategies. Let D be the set of all direct anonymous mechanisms equipped with some natural σ-algebra $\mathcal{D}$. A seller's strategy is a probability measure δ^s on $(D, \mathcal{D})$. A buyer needs to decide on whether or not to participate in the market, and if he does, which sellers (who are characterized by the mechanisms they post) to visit. To acknowledge that a buyer's strategy depends (only) on his value x_k and the fact that—due to the lack of coordination—buyers treat all sellers who post the same mechanism symmetrically, we denote his strategy by δ_k^b, a measure on $(D, \mathcal{D})$. If $\delta_k^b(D) < 1$, then buyers with value x_k will choose not to participate in the market with probability $1 - \delta_k^b(D)$, in which case their payoff will be zero.[16] Since a buyer can only visit a mechanism if a seller posted it, we require that for each $k = 1, 2$, the measure δ_k^b is absolutely continuous with respect to δ^s.[17] The Radon-Nikodym derivative

[12] In line with most of the literature, we abstract from mechanisms that condition on other mechanisms present in the market. See Epstein and Peters (1999) and Peters (2001) for a detailed discussion.

[13] This assumes, for simplicity, that a positive measure of buyers and sellers visit the submarket. If this is not the case, we can use Radon-Nykodym derivatives to define queue lengths.

[14] The assumption that P_n only depends on the queue length and not its composition is a natural benchmark since it creates a distinction between meetings and matches.

[15] It is straightforward to allow buyers to observe only a fraction of the sellers. If the fraction of sellers that a buyer observes is type independent, this will not change our results.

[16] The assumption that all sellers post a mechanism is without loss of generality, because they can stay inactive by posting a sufficiently inattractive mechanism, e.g. a reserve price above x_2.

[17] This rules out the scenario in which a zero measure of sellers attracts a positive measure of buyers. This restriction is natural and can be justified by the optimal choices of buyers and sellers (see below).

$d\delta_k^b/d\delta^s$ determines the queue length and queue composition—i.e., how many buyers and what types of buyers are available per seller—for each mechanism (almost surely) in the support of δ^s. Formally, for (almost every) mechanism ω in the support of δ^s and $k = 1, 2$, the queue length of buyers with value x_k, $q_k(\omega)$, is given by

$$q_k(\omega) = B_k \frac{d\delta_k^b}{d\delta^s}. \tag{1}$$

Payoffs. Note that for any mechanism $\omega \in D$, the expected payoff of a seller who posts mechanism ω is completely determined by ω and its queue $\boldsymbol{q}(\omega) \equiv (q_1(\omega), q_2(\omega))$. Therefore, we can denote it by $\pi(\omega, \boldsymbol{q}(\omega))$. Similarly, let $V_k(\omega, \boldsymbol{q}(\omega))$ denote the expected payoff of a buyer with value x_k from visiting a submarket with mechanism ω which has queue $\boldsymbol{q}(\omega)$.

Market Utility and Beliefs. We now define conditions on buyers' and sellers' strategy $(\delta^s, \delta_1^b, \delta_2^b)$ which need to be satisfied in equilibrium. First, consider the optimality of buyers' strategies. The *market utility function* U_k is defined to be the maximum utility that a buyer with value x_k can obtain by visiting a seller or being inactive.

$$U_k = \max \left(\max_{\omega \in \text{supp}(\delta^s)} V_k(\omega, \boldsymbol{q}(\omega)), 0 \right),$$

where $\boldsymbol{q}(\omega)$ is given by equation (1). Of course, optimality of buyers' choices requires that buyers choose the mechanism that yields the highest payoff. Formally, we have

$$V_k(\omega, \boldsymbol{q}(\omega)) \leq U_k \quad \text{with equality if } \omega \text{ is in the support of } \delta_k^b.$$

Next, we consider the optimality of sellers' strategies. All posted mechanisms should generate the same expected payoff π^* and there should be no profitable deviations. A seller considering a deviation to a mechanism $\widetilde{\omega}$ not in the support of δ^s needs to form beliefs regarding the queue $\boldsymbol{q}(\widetilde{\omega})$ that he will be able to attract. We call a queue $\boldsymbol{q}(\widetilde{\omega})$ *compatible* with the mechanism $\widetilde{w}$ and the market utility function U_k if for any $k \in \{1, 2\}$,

$$V_k(\widetilde{\omega}, \boldsymbol{q}(\widetilde{\omega})) \leq U_k \quad \text{with equality if } q_k(\widetilde{\omega}) > 0. \tag{2}$$

Of course, for any mechanism ω in the support of δ^s, $\boldsymbol{q}(\omega)$ is compatible with mechanism ω and the market utility function because of the optimal search behavior of buyers. The literature usually assumes that when posting $\widetilde{w}$, the seller will expect the most favorable queue among all queues that are compatible with $\widetilde{\omega}$ and the market utility function (see, for example, McAfee, 1993; Eeckhout and Kircher, 2010a,b). That is,

$$\boldsymbol{q}(\widetilde{\omega}) = \arg\max_{\widetilde{\boldsymbol{q}}} \pi(\widetilde{\omega}, \widetilde{\boldsymbol{q}}) \tag{3}$$

where the choice of $\widetilde{\boldsymbol{q}}$ is subject to the constraint in equation (2).[18] Initially, we will adopt this convention, but later we will show that—with some mild restrictions on the meeting technology—this assumption is unnecessary: when $\widetilde{\omega}$ is (without loss of generality) a second-price auction with reserve price or entry fee, these restrictions imply that there is only one possible queue compatible with $\widetilde{\omega}$ and the market utility function.

Equilibrium Definition. We can now define an equilibrium as follows.

Definition 1. *A directed search equilibrium is a tuple $(\delta^s, \delta_1^b, \delta_2^b)$ of strategies with the following properties:*

1. *Each ω in the support of δ^s maximizes $\pi(\omega, \boldsymbol{q}(\omega))$, where, depending on whether or not ω belongs to the support of δ^s, $\boldsymbol{q}(\omega)$ is given by equations (1) and (3), respectively.*
2. *For each buyer type x_k, δ_k^b is absolutely continuous with respect to δ^s. If $\delta_k^b(D) > 0$, every ω in the support of δ_k^b maximizes $V_k(\omega, \boldsymbol{q}(\omega))$. If $\delta_k^b(D) = 0$, then for any mechanism ω in the support of δ^s the buyer value $V_k(\omega, \boldsymbol{q}(\omega))$ is non-positive.*
3. *Aggregating queues across sellers does not exceed the total measure of buyers of each type. That is, $\int q_k(\omega) d\delta^s(w) \leq B_k$ for each $k \in \{1, 2\}$.*

2.2 Alternative Representation of Meetings

We first present a transformation of the meeting technology that greatly simplifies the analysis. In particular, we introduce a new function $\phi(\mu, \lambda)$ with $0 \leq \mu \leq \lambda$, defined as

$$\phi(\mu, \lambda) = 1 - \sum_{n=0}^{\infty} P_n(\lambda) \left(1 - \frac{\mu}{\lambda}\right)^n. \tag{4}$$

To understand this function, consider a submarket in which sellers face a queue length λ. Suppose that a fraction μ/λ of the buyers in the submarket has the high value x_2. Since the meeting technology treats different buyers symmetrically, $\phi(\mu, \lambda)$ then represents the probability that a seller meets at least one high-value buyer.

[18] For some mechanism $\widetilde{w}$ there may not exist a compatible queue because $\widetilde{w}$ is either too attractive or too unattractive. If $\widetilde{w}$ is too unattractive, we can set $\widetilde{\boldsymbol{q}}$ to be the zero vector. In Section 4, we show that sellers can not do better than posting a second-price auction with a reserve price, which implies that $\widetilde{w}$ will not be too attractive in the above sense.

The function $\phi(\mu, \lambda)$ allows us to study competing mechanisms with general meeting technologies in a way that is both more tractable and more intuitive than with $P_n(\lambda)$, $n = 0, 1, \ldots$. The following Proposition establishes that no information is lost by considering $\phi(\mu, \lambda)$ instead of $P_n(\lambda)$, since we can always recover one from the other.

Proposition 1. *If $\phi(\mu, \lambda)$ is generated by some $\{P_n(\lambda) : n = 0, 1, 2, \ldots\}$, then we can recover $P_n(\lambda)$ from $\phi(\mu, \lambda)$ by*

$$P_n(\lambda) = \frac{(-\lambda)^n}{n!} \frac{\partial^n}{\partial \mu^n} (1 - \phi(\mu, \lambda))\bigg|_{\mu=\lambda}. \tag{5}$$

Proof. See Appendix A.1. □

To develop intuition for $\phi(\mu, \lambda)$, suppose that $\Delta\lambda$ more buyers visit this submarket, then the probability that the seller meets at least one *incumbent* high-value buyer becomes $\phi(\mu, \lambda + \Delta\lambda)$, where μ is the measure of the incumbent high-value buyers. Therefore, $\phi_\lambda(\mu, \lambda) \equiv \partial\phi(\mu, \lambda)/\partial\lambda$ measures the effect of the new entrants on the meeting probabilities between sellers and incumbent high-value buyers: $\phi_\lambda(\mu, \lambda) < 0$ (resp. > 0) represents negative (resp. positive) meeting externalities. In the special case of $\phi_\lambda(\mu, \lambda) = 0$, there are no meeting externalities among buyers.

For future reference, note that

$$\phi_\mu(\mu, \lambda) \equiv \frac{\partial\phi(\mu, \lambda)}{\partial\mu} = \sum_{n=1}^{\infty} Q_n(\lambda) \left(1 - \frac{\mu}{\lambda}\right)^{n-1}. \tag{6}$$

That is, $\phi_\mu(\mu, \lambda)$ is the probability for a buyer to be part of a meeting in which all other buyers (if any) have low valuations. In this case, a high-type buyer increases social surplus *directly*, since the good would have been allocated to a low-type buyer in his absence. In a second-price auction, this is also the probability that a high-type buyer wins the auction with strictly positive payoff, which we define to be the *winning probability* of high-type buyers.[19] Since for each n, $(1 - \mu/\lambda)^{n-1}$ is decreasing in μ, $\phi_\mu(\mu, \lambda)$ is then also decreasing in μ, implying that $\phi(\mu, \lambda)$ is concave in μ, which holds strictly if and only if $P_0(\lambda) + P_1(\lambda) < 1$.[20]

[19] Because buyers types are discrete, buyers' winning and trading probability are different: a buyer may compete with another buyer with the same value. But as we will see later, this difference is not important for our analysis. The use of this winning probability is a canonical technique developed by McAfee (1993) and Peters and Severinov (1997). They show that buyers' winning probability must be equal at competing sellers. Our function $\phi(\mu, \lambda)$ incorporates their approach by its first partial derivative $\phi_\mu(\mu, \lambda)$ and does more because its second partial derivative $\phi_\lambda(\mu, \lambda)$ represents meeting externalities. Moreover, the function $\phi(\mu, \lambda)$ itself is intimately linked with surplus. See Lemma 1 and also Lemma 3 for the formal statements.

[20] For each $n \geq 0$, $-(1 - \mu/\lambda)^n$ is increasing and concave in μ, and it is strictly concave in μ if and only if $n \geq 2$. Therefore, $\phi(\mu, \lambda)$ is strictly concave in μ if and only if there exists at least one $n \geq 2$ such that $P_n(\lambda) > 0$.

Two special cases of equation (6) are worth mentioning: i) $\phi_\mu(0,\lambda) = 1 - Q_0(\lambda)$, i.e. the probability that a buyer meets a seller, and ii) $\phi_\mu(\lambda,\lambda) = Q_1(\lambda)$, i.e. the probability that a buyer meets a seller without other buyers.

Examples of Meeting Technologies.

1. *Bilateral.* With bilateral meeting technologies, each seller meets at most one buyer, i.e., $P_0(\lambda) + P_1(\lambda) = 1$ with $P_1(\lambda)$ strictly concave. In this case, $\phi(\mu,\lambda) = P_1(\lambda)\mu/\lambda$.

2. *Invariant (e.g. urn-ball or geometric).* Invariant meeting technologies are defined by the absence of meeting externalities, i.e. $\phi_\lambda(\mu,\lambda) = 0$ for any $0 \le \mu \le \lambda$.[21] One example is the urn-ball technology, which specifies that the number of buyers meeting a seller follows a Poisson distribution with a mean equal to the queue length λ. That is, $P_n(\lambda) = e^{-\lambda}\lambda^n/n!$ or $\phi(\mu,\lambda) = 1 - e^{-\mu}$. A second example is the geometric technology of Lester et al. (2015), where agents in a submarket are uniformly positioned on a circle and buyers walk clockwise to the nearest seller. This yields $P_n(\lambda) = \left(\frac{1}{1+\lambda}\right)\left(\frac{\lambda}{1+\lambda}\right)^n$ or $\phi(\mu,\lambda) = \mu/(1+\mu)$.

3. *Geometric truncated at* 2. The number of buyers that a seller meets follows a geometric but the maximum number of meetings is 2. Thus $P_n(\lambda) = \left(\frac{1}{1+\lambda}\right)\left(\frac{\lambda}{1+\lambda}\right)^n$ for $n < 2$, and $P_2(\lambda) = 1 - P_0(\lambda) - P_1(\lambda)$. Appendix B.4.1 fully characterizes this case.

4. *Geometrically truncated geometric.* This meeting technology is similar to the geometric technology, except sellers may face time or capacity constraints, stopping them from meeting all buyers that try to visit them. The maximum number of buyers that a seller can meet follows a geometric distribution with parameter σ and support $\{1, 2, 3, \dots\}$. The number of meetings taking place is therefore the minimum of two geometric variables, the number of buyers that try to meet/contact the seller and the seller's capacity. This technology reduces to a bilateral one when $\sigma = 0$ and to an invariant one when $\sigma = 1$.[22] This technology satisfies all assumptions in this paper, including the ones in Online Appendix, and is analyzed in detail in Appendix B.4.2, which shows that for intermediate σ, there will be two submarkets in equilibrium: one submarket contains all high-type buyers and some low-type buyers and the other submarket contains the remaining low-type buyers. This phenomenon of partial segmentation is new to the literature.

[21] Lester et al. (2015) first introduced invariant meeting technologies in terms of $P_n(\lambda)$. Cai et al. (2017) show that their definition is equivalent to $\phi_\lambda(\mu,\lambda) = 0$.

[22] For a similar technology with Poisson applications, see Wolthoff (2018).

Assumptions. In most of our analysis, we will remain agnostic about the exact meeting technology and just make a few weak assumptions regarding $\phi(\mu, \lambda)$ which provide the minimal structure we need to prove our results. Two assumptions are necessary for most of our results, so we present them here; the remaining assumptions are introduced just before the specific results that require them. The meeting technologies that we presented above satisfy the assumptions that we introduce below. To introduce the assumptions, we first apply a change of notation and define $z = \mu/\lambda$ as the fraction of high-type buyers in the queue. Further, to simplify notation, we define $m(\lambda) \equiv \phi(\lambda, \lambda)$, which is the probability that a seller meets at least one buyer.

Assumption 1. *$\phi(\lambda z, \lambda)$ is strictly concave in λ for any $z \in (0, 1]$. Furthermore, $\lim_{\lambda \to 0} m'(\lambda) = 1$ and $\lim_{\lambda \to \infty} m(\lambda) - \lambda m'(\lambda) = 1$.*

This assumption states that if we hold the fraction of high-type buyers constant, the marginal effect of an extra buyer on the seller's probability of meeting at least one high-type is decreasing in the total queue length. Since $m(\lambda) \equiv \phi(\lambda, \lambda)$, this assumption also implies that $m(\lambda)$ is strictly concave. The second part of Assumption 1 is more a normalization than an assumption. It implies that in a submarket with only buyers with valuation x_k, the marginal contribution of these buyers is x_k when $\lambda \to 0$, while for sellers it is x_k when $\lambda \to \infty$.

Our second assumption concerns the probability $\phi_\mu(\lambda z, \lambda)$ that a high-type buyer increases surplus directly—i.e., faces no competition from other high-type buyers. We assume that this probability decreases if we add more buyers to the queue, holding the fraction of high-type buyers constant at z.

Assumption 2. *$\phi_\mu(\lambda z, \lambda)$ is strictly decreasing in λ for $0 \leq z \leq 1$.*

As mentioned after equation (6), $\phi_\mu(\lambda, \lambda) = Q_1(\lambda)$ and $\phi_\mu(0, \lambda) = 1 - Q_0(\lambda)$. Thus, Assumption 2 implies that (i) in submarkets with longer queues, it is less likely that a buyer turns out to be the only one present, and (ii) buyers are less likely to meet a seller if the queue length in the submarket increases, which could be interpreted as a form of congestion.

3 Social Planner

3.1 Surplus and Planner's Problem

Surplus. We start our analysis with the following lemma which derives total surplus and agents' marginal contribution to this surplus in a submarket with queue $\boldsymbol{q}$. To use our alternative representation of meeting technologies, we apply a change of notation and, as before, define μ as the queue length of buyers with value x_2 and λ as the total queue length, i.e., $\mu = q_2$ and $\lambda = q_1 + q_2$.

Lemma 1. *Consider a submarket with a measure 1 of sellers and a queue (μ, λ) of buyers. Total surplus in the submarket then equals*

$$S(\mu, \lambda) = m(\lambda) + (x_2 - 1)\,\phi(\mu, \lambda) \tag{7}$$

The marginal contribution to surplus of low-type and high-type buyers are, respectively,

$$T_1(\mu, \lambda) = m'(\lambda) + (x_2 - 1)\,\phi_\lambda(\mu, \lambda) \tag{8}$$

$$T_2(\mu, \lambda) = m'(\lambda) + (x_2 - 1)\,(\phi_\mu(\mu, \lambda) + \phi_\lambda(\mu, \lambda))\,. \tag{9}$$

A seller's marginal contribution to surplus equals

$$R(\mu, \lambda) = m(\lambda) - \lambda m'(\lambda) + (x_2 - 1)\,(\phi(\mu, \lambda) - \mu\phi_\mu(\mu, \lambda) - \lambda\phi_\lambda(\mu, \lambda))\,. \tag{10}$$

Proof. See below and Lemma 3 for the general case with N buyer types. □

The first term in equation (7) accounts for the fact that a surplus of (at least) 1 is generated whenever a seller meets at least one buyer. The second term captures that an additional surplus of $x_2 - 1$ is realized when a seller meets at least one high-type buyer.

To understand (8), note that $T_1(\mu, \lambda) = S_\lambda(\mu, \lambda)$ since adding a low-type buyer to the submarket increases λ but has no effect on μ. The first term of (8) reflects the effect of the extra buyer on the number of matches, while the second term represents the externalities that he may impose on meetings between sellers and high-type buyers. Since $m'(\lambda) = \phi_\mu(\lambda, \lambda) + \phi_\lambda(\lambda, \lambda)$, equation (8) can also be written as

$$T_1(\mu, \lambda) = \phi_\mu(\lambda, \lambda) + \phi_\lambda(\lambda, \lambda) + (x_2 - 1)\,\phi_\lambda(\mu, \lambda),$$

where the first term describes the buyer's direct contribution to surplus which arises when there are no other buyers, as discussed below equation (6). The second term and third term represent the externalities that the buyer may impose on sellers' meetings with, respectively, other low-type and high-type buyers.

To understand (9), note that $T_2(\mu, \lambda) = S_\mu(\mu, \lambda) + S_\lambda(\mu, \lambda)$ since adding an additional high-type buyer to the submarket increases both μ and λ. Therefore, $T_2(\mu, \lambda) = T_1(\mu, \lambda) + (x_2 - 1)\,\phi_\mu(\mu, \lambda)$. That is, the additional high-type buyer creates the same meeting externalities as an extra low-type buyer, but creates additional surplus when there are no other buyers or only low-type buyers, which happens with probability $\phi_\mu(\mu, \lambda)$.

Finally, to understand equation (10), define $z = \mu/\lambda$ to be the fraction of high-type buyers in the queue. If we add λ more buyers to the submarket while keeping z fixed, then adding one more seller increases surplus by $S(\lambda z, \lambda)$. Therefore, $R(\lambda z, \lambda) = S(\lambda z, \lambda) - \lambda \frac{\partial S(\lambda z, \lambda)}{\partial \lambda}$, or, alternatively $R(\mu, \lambda) = S(\mu, \lambda) - \mu T_2(\mu, \lambda) - (\lambda - \mu) T_1(\mu, \lambda)$.

Planner's Problem. One can think of the planner's problem as a three-step optimization problem: first, the planner chooses the number of submarkets to open; second, he determines the allocation of buyers and sellers to the different submarkets; third, he decides on the allocation of the good after meetings have taken place. The third step is trivial: at each seller, the good is always allocated to the buyer with the highest valuation. The first two steps will depend on the meeting technology and the distribution of valuations. Suppose that the planner creates L submarkets with positive seller measures $\alpha^1, \ldots, \alpha^L$, respectively, and potentially an additional submarket with no sellers but only buyers. Of course, this additional submarket generates no surplus but could be useful for reducing meeting externalities. The queue in submarket $\ell = 1, \ldots, L$ is (μ^i, λ^i). The planner's problem is thus

$$\mathcal{S}^*(B_1, B_2) = \sup_{L \geq 1} \sup_{\{(\alpha^\ell, \mu^\ell, \lambda^\ell) \,|\, \ell = 1, \ldots, L\}} \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell) \tag{11}$$

subject to the standard accounting constraints $\sum_{\ell=1}^L \alpha^\ell = 1$, $\sum_{\ell=1}^L \alpha^\ell \mu^\ell \leq B_2$, and $\sum_{\ell=1}^L \alpha^\ell (\lambda^\ell - \mu^\ell) \leq B_1$.[23]

3.2 Characterization of the Planner's Solution

In this section, we characterize the planner's solution. Since the case of bilateral meeting technologies is well understood and to avoid the issue of division by zero, we impose the restriction that $P_0(\lambda) + P_1(\lambda) < 1$ for any $\lambda > 0$ so that $\phi_{\mu\mu}(\mu, \lambda)$ is never zero.

Number of Submarkets. It is not clear a priori that there is an upper bound on the number of submarkets L for any endowment of buyers (B_1, B_2). However, the following Proposition shows that such an upper bound exists. To state the result, we define an *idle* submarket as a market that either contains only buyers or only sellers (as opposed to an *active* submarket in which both buyers and sellers are present).[24]

Proposition 2. *When there are two buyer types, the planner's problem can be solved by opening at most three submarkets, including one potentially idle submarket.*

Proof. See Appendix A.2. □

[23] The inequalities reflect that the planner may require some buyers to be inactive and not visit any seller.
[24] The planner will of course never simultaneously choose an idle market for buyers and one for sellers.

Proposition 2 serves two goals. First, it is a technical result on existence in the sense that it establishes that the supremum of surplus over all possible allocations can indeed be reached as a maximum. Second, it limits the complexity of the planner's problem by bounding the number of submarkets. The intuition is as follows. By equation (11), total surplus is a convex combination of the surpluses generated by individual submarkets. The planner chooses the number of submarkets to find the maximum value that such convex combinations can reach, which simply corresponds to finding the concave hull of the surplus function S as presented in equation (7). As a result of this correspondence, the Fenchel-Bunt Theorem provides an upper bound for the number of submarkets needed to solve the planner's problem.[25]

Concavity. If the surplus function $S(\mu, \lambda)$ is jointly concave in (μ, λ) then the concave hull is of course S itself. In this case, merging any two submarkets always increases total surplus and the planner's solution is simply to pool all buyers and sellers into a single submarket (see Cai et al., 2017). However, as we will show below, joint concavity is often violated. In these cases, the planner needs to solve a non-concave optimization problem, which is notoriously difficult. We make progress below by formulating weak restrictions on the meeting technology. Under those restrictions, the first-order conditions are both necessary and sufficient, and a simple algorithm will solve the planner's problem.

Even if concavity of $S(\mu, \lambda)$ fails globally, it still needs to hold locally in any submarket (μ, λ) satisfying $0 < \mu < \lambda$. Otherwise, by definition, we can break the submarket into two and reallocate sufficiently small measures of buyers $\Delta\mu$ and $\Delta\lambda$ to increase total surplus, i.e.

$$\frac{1}{2}S(\mu - \Delta\mu, \lambda - \Delta\lambda) + \frac{1}{2}S(\mu + \Delta\mu, \lambda + \Delta\lambda) > S(\mu, \lambda).$$

The Hessian matrix of the surplus function $S(\mu, \lambda)$ must therefore be negative semi-definite at the point (μ, λ). This normally requires two inequalities to hold since the Hessian is a 2×2 matrix. However, the surplus function $S(\mu, \lambda)$ is linear in $\phi(\mu, \lambda)$, which in turn is always concave in μ. The only remaining condition therefore is that the determinant of the Hessian is positive, such that we have the following result.

Lemma 2. *A submarket (μ, λ) with $0 < \mu < \lambda$ can be part of the planner's solution only if the Hessian matrix of the surplus function (7) is negative semi-definite at (μ, λ), which is*

[25] The classical Caratheodory theory states that any point in the convex hull of a set $A \subset \mathbb{R}^n$ can be represented as a convex combination of $n + 1$ points of A. The Fenchel-Bunt Theorem states that if the set A is connected, then for the above construction we only need n points instead of $n + 1$. Since the graph of $S : \mathbb{R}^2 \to \mathbb{R}$ is a connected subset in $\mathbb{R}^3$, the Fenchel-Bunt Theorem implies that we only need three points to construct the concave hull of S.

equivalent to:

$$H(\mu,\lambda) \equiv \frac{1}{-m''(\lambda)} \left(\phi_{\lambda\lambda}(\mu,\lambda) - \frac{\phi_{\mu\lambda}(\mu,\lambda)^2}{\phi_{\mu\mu}(\mu,\lambda)} \right) \leq \frac{1}{x_2 - 1} \tag{12}$$

Proof. See Appendix A.3. □

This condition is automatically satisfied if $\phi_{\mu\mu}\phi_{\lambda\lambda} - \phi_{\mu\lambda}^2 \geq 0$, i.e. joint concavity of the meeting technology therefore implies joint concavity of the surplus function (see Cai et al., 2017). However, the second-order condition (12) may hold even if the meeting technology $\phi(\mu,\lambda)$ is non-concave everywhere, as is for example the case for the geometrically truncated geometric technology with $\sigma < 1$.[26] To understand this, note that the expression for surplus $S(\mu,\lambda)$ in equation (7) has two terms. If $\phi(\mu,\lambda)$ is non-concave, the second term in (7) provides a force against pooling; the factor $\phi_{\lambda\lambda} - \phi_{\mu\lambda}^2/\phi_{\mu\mu}$ in (12) measures its strength. However, because sellers' probability to meet at least one buyer is concave, the first term in (7) is always a force for pooling, with $-m''(\lambda)$ in (12) representing its strength.[27]

Participation. The expressions for the marginal contribution to surplus of buyers and sellers in lemma 1 allow us to address the planner's participation decisions. Intuitively, all high-type buyers must always all participate. If a subset of high-type buyers were idle, then all low-type buyers would have to be idle as well because they create less surplus. However, with only high-type buyers present, adding high-type buyers always increases surplus since $m(\lambda)$ is strictly increasing. If Assumption 1 holds, the marginal contribution of sellers in a submarket is always positive so they should all be active. Lemma 4 in Appendix A.4 formalizes this and derives conditions for the planner under which either all buyers or all sellers should be assigned to *active submarkets* (i.e. submarkets that contain both buyers and sellers).

Level Curves. The next step in our analysis is to consider how conditions that the planner's solution must satisfy constrain the submarkets $(\lambda z, \lambda)$ that may be formed. One such condition is that sellers' contribution to surplus must be equal in all active submarkets created by the planner. These submarkets must therefore lie on some level curve $R(\lambda z, \lambda) = R^*$. Intuitively, sellers' marginal contribution to surplus is higher if there are either more buyers (holding z constant) or more high-type buyers (holding λ constant). Thus the level curves

[26] It is easy to verify that $\phi_{\mu\mu}\phi_{\lambda\lambda} - \phi_{\mu\lambda}^2 = -(1-\sigma)^2/(1+\sigma\mu+(1-\sigma)\lambda)^4 < 0$ in this case.

[27] To see this, suppose that we divide a submarket (μ,λ) into two submarkets $(\mu-\Delta\mu, \lambda-\Delta\lambda)$ and $(\mu+\Delta\mu, \lambda+\Delta\lambda)$ with $\Delta\lambda > 0$ and $\Delta\mu$ of indeterminate sign. Then, the loss of total surplus stemming from the first part of the surplus function is $m(\lambda) - \frac{1}{2}m(\lambda+\Delta\lambda) - \frac{1}{2}m(\lambda-\Delta\lambda)$, which equals $-m''(\lambda)\Delta\lambda^2 > 0$. The surplus gain from the second part of the surplus function is $(x_2-1)(\phi_{\mu\mu}\Delta\mu^2 + 2\phi_{\mu\lambda}\Delta\mu\Delta\lambda + \phi_{\lambda\lambda}\Delta\lambda^2)/2$. The gain is maximized when $\Delta\mu = -\Delta\lambda\phi_{\mu\lambda}/\phi_{\mu\mu}$, and the maximal gain is $(x_2-1)(\phi_{\lambda\lambda} - \phi_{\mu\lambda}^2/\phi_{\mu\mu})$.

of $R(\lambda z, \lambda)$ are downward sloping in the λ-z plane, as we prove formally in Lemma 6 of Appendix A.4. Figure 1 illustrates two downward-sloping level curves of $R(\lambda z, \lambda)$.

A further requirement on the planner's solution is that buyers' marginal contribution to surplus must be equal in all submarkets that they visit. This requires knowledge of how a buyer's marginal contribution $T_1(\lambda z, \lambda)$ or $T_2(\lambda z, \lambda)$ varies along level curves of $R(\lambda z, \lambda)$. It turns out that the answer depends on the *sign* of the determinant of the Hessian of the surplus function. In Figure 1, the red solid curve is where the determinant is zero; the area to its left is where the determinant is negative and the area to its right is where the determinant is positive. Lemma 7 in Appendix A.4 shows that along a level curve of $R(\lambda z, \lambda)$, high-type buyers' marginal contribution $T_2(\lambda z, \lambda)$ is strictly decreasing in z in the segment to the left of the red solid curve (e.g., segment S_0S_1) and strictly increasing in z in the segment to the right of the red solid curve (e.g., segment S_1S_2). The reverse holds for $T_1(\lambda z, \lambda)$.

Therefore, each level curve of $R(\lambda z, \lambda)$ can be divided into two intervals, and buyers' marginal contributions, $T_1(\lambda z, \lambda)$ and $T_2(\lambda z, \lambda)$, vary monotonically within each interval. However, inspecting Figure 1 shows that this requires that the level curve of $R(\lambda z, \lambda)$ intersects with the red curve only once and from the left. For this to hold in general, we require one additional (weak) assumption on the meeting technology.

Assumption 3 (Single Crossing)**.** *At any point* (z, λ) *where* $H(\lambda z, \lambda) > 0$, *we have* $\partial H(\lambda z, \lambda)/\partial \lambda > 0$ *and*

$$-\frac{\partial \phi_\mu(\lambda z, \lambda)/\partial z}{\partial \phi_\mu(\lambda z, \lambda)/\partial \lambda} < -\frac{\partial H(\lambda z, \lambda)/\partial z}{\partial H(\lambda z, \lambda)/\partial \lambda}. \tag{13}$$

It is worth highlighting that—like our other assumptions—Assumption 3 concerns the meeting technology only.[28] The left-hand side (resp. right-hand) of (13) denotes the slope of the level curve of $\phi_\mu(\lambda z, \lambda)$ (resp. $H(\lambda z, \lambda)$) in the z-λ plane. Thus, Assumption 3 states that any level curve of $\phi_\mu(\lambda z, \lambda)$ crosses any positive level curve of $H(\lambda z, \lambda)$ at most once and from left. Perhaps surprisingly, it also implies that each level curve of $R(\lambda z, \lambda)$ crosses the curve $H(\lambda z, \lambda) = 1/(x_2 - 1)$ (where the determinant of the Hessian matrix is zero) at most once and from left. This claim is made precise in Lemma 8 in Appendix A.4.

Two Submarkets. We can now further tighten the bound on the number of submarkets. We illustrate the argument in Figure 1. Suppose that at the social optimum, the marginal contribution to surplus of sellers is R^* and the black dashed level curve $R(\lambda z, \lambda) = R^*$ intersects the red curve $H(\lambda z, \lambda) = 1/(x_2 - 1)$ at point $S_1 = (\lambda^* z^*, \lambda^*)$. The first part of

[28] By the definition in equation (12), $H(\lambda z, \lambda) > 0$ if and only if $\phi_{\mu\mu}\phi_{\lambda\lambda} - \phi_{\mu\lambda}^2 < 0$ at the point (z, λ). Hence, the above assumption becomes void and is satisfied automatically for jointly concave meeting technologies (see Cai et al., 2017). In this paper, we consider the more realistic case where ϕ is not always or never concave in (μ, λ).

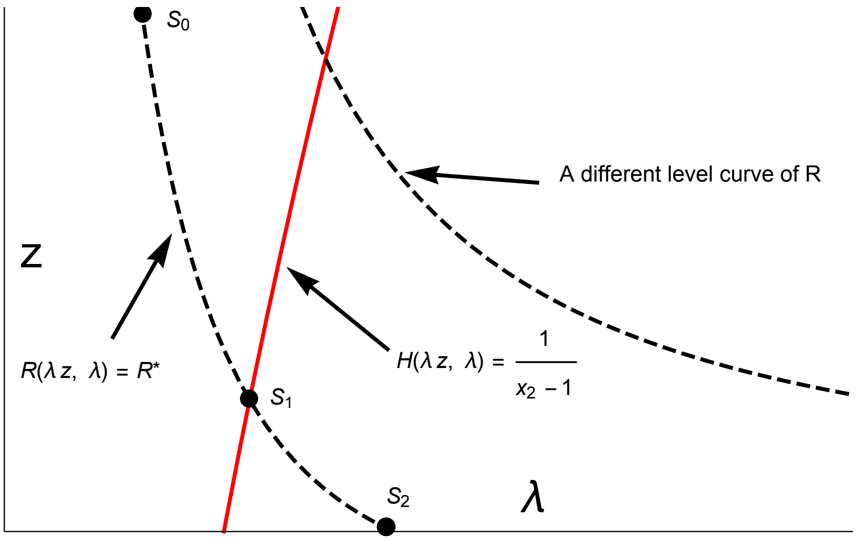


Figure 1: Illustration of Assumption 3 (Single Crossing)

Assumption 3 ensures that the second-order condition (12) is satisfied left of the red line and violated right of the red line. In other words, submarkets with $0 < z < z^*$ (i.e., points on the S_1S_2 trajectory) cannot be part of the planner's solution. The only feasible submarket on this side is therefore the corner S_2 where $z^* = 0$.

In contrast, the second-order condition is satisfied in submarkets with $z \geq z^*$, i.e. points on the S_0S_1 trajectory. However, by Lemma 7, $T_2(\lambda z, \lambda)$ is strictly decreasing in z along this trajectory. Since the marginal contribution of high-type buyers must be the same among all submarkets containing such buyers, there can therefore only exist one submarket with $z \geq z^*$. To sum up, there exist at most two submarkets in the social optimum: one with $z \geq z^*$ and one with $z = 0$.

The above observation greatly simplifies the analysis because it implies that there are only three possible solutions: (i) complete pooling, i.e. all agents are in one market; (ii) complete separation, i.e. there is one submarket for all high-type buyers and one (possibly idle) for all low-type buyers; (iii) mixing, i.e. there is one submarket that contains all high-type and some low-type buyers, and one (possibly idle) submarket with the remaining low-type buyers. From Cai et al. (2017) , we know that invariant—or more generally, jointly concave—technologies imply complete pooling, while bilateral technologies imply complete separation. The third possibility, which spans the range between these extremes, is new. It allows the planner to take advantage of multilateral meetings to screen ex post by pooling high types with some low types, while reducing the degree of crowding out by separating other low types.

The optimal extent of separation depends on the magnitude of the meeting externalities, the measures of high and low types, and the dispersion in valuations. To solve for it, assume, without loss of generality, that the planner opens two submarkets, one with a high average valuation containing all high-type buyers, and one with a low average valuation without high-type buyers. Two decisions then remain: i) how to allocate low-type buyers and ii) how to allocate sellers. We solve these decisions sequentially.

Allocation of Sellers. Suppose the planner assigns a measure b_1 of low-type buyers to the submarket with the high average valuation. The optimal allocation of sellers, denoted by $\alpha^*(b_1)$, then solves

$$\overline{S}(b_1) = \max_{\alpha} \quad \alpha S\left(\frac{B_2}{\alpha}, \frac{B_2 + b_1}{\alpha}\right) + (1-\alpha) S\left(0, \frac{B_1 - b_1}{1-\alpha}\right). \tag{14}$$

Of course, $\alpha^*(B_1) = 1$. For $b_1 < B_1$, both terms on the right-hand side are concave in α by Lemma 5 in Appendix A.4, such that $\alpha^*(b_1)$ is uniquely characterized by the first-order condition, i.e.

$$\begin{aligned} \alpha^* &= 1 && \text{if } R(B_2, B_2 + b_1) \geq 1 \\ R\left(\frac{B_2}{\alpha^*}, \frac{B_2 + b_1}{\alpha^*}\right) &= R\left(0, \frac{B_1 - b_1}{1-\alpha^*}\right) && \text{if } R(B_2, B_2 + b_1) < 1 \end{aligned} \tag{15}$$

The first case in (15) describes a corner solution, where sellers' marginal contribution to surplus is higher in the submarket with the high average valuation even when all sellers are allocated to this submarket. The second case describes an interior solution, where sellers' marginal contributions must be the same across the two submarkets. Note that $\alpha^*(b_1)$ can never be 0, because the planner will never leave high-type buyers idle.

Allocation of Low-Type Buyers. Having solved for $\alpha^*(b_1)$, we next consider the allocation of low-type buyers. By the envelope theorem, $\overline{S}(b_1)$ is differentiable and for $b_1 < B_1$,

$$\overline{S}'(b_1) = T_1\left(\frac{B_2}{\alpha^*(b_1)}, \frac{B_2 + b_1}{\alpha^*(b_1)}\right) - T_1\left(0, \frac{B_1 - b_1}{1-\alpha^*(b_1)}\right). \tag{16}$$

That is, the additional surplus generated by moving one low-type buyer from the low-average-valution to the high-average-valuation submarket is simply the difference between the buyer's marginal contributions to surplus in the two submarkets.

The special case $b_1 = B_1$ warrants discussion as it is not defined by the above equation (because it leads to 0/0 in the final argument). In this case, the planner allocates all sellers and buyers to the submarket with the high average valuation, and considers the welfare effect of moving an ε number of low-type buyers to a separate submarket. Whether the planner also moves sellers to this separate submarket depends on $R(B_2, B_2 + B_1)$. In particular, if $R(B_2, B_2 + B_1) \geq 1$, then the planner will keep all sellers in the submarket with the high average valuation, because sellers' contribution to surplus in the second submarket is bounded by 1; in contrast, if $R(B_2, B_2 + B_1) < 1$, then the planner will move a small number of sellers to the separate submarket to equalize sellers' contribution to surplus across the two submarkets. Regardless,

$$\overline{S}'(B_1) = T_1(B_2, B_2 + B_1) - T_1(0, \lambda) \tag{17}$$

where λ is such that $R(B_2, B_2 + B_1) = R(0, \lambda)$ if $R(B_2, B_2 + B_1) < 1$, otherwise $\lambda = \infty$.

The optimal b_1, b_1^*, must then satisfy the first-order condition, i.e.

$$\overline{S}'(b_1^*) \leq 0 \text{ if } b_1^* = 0; \quad \overline{S}'(b_1^*) = 0 \text{ if } 0 < b_1^* < B_1; \quad \overline{S}'(b_1^*) \geq 0 \text{ if } b_1^* = B_1. \tag{18}$$

It turns out that the first-order condition is sufficient even though $\overline{S}(b_1)$ is not necessarily concave. The following proposition formalizes our results.[29]

Proposition 3. *Under Assumption 1, 2, and 3, at the social optimum, all sellers are active, and there are at most two submarkets, one of which contains all high-type buyers and has a shorter queue. Furthermore, the planner's solution is unique, and the first-order conditions (15) and (18) are necessary and sufficient.*

Proof. See Appendix A.5. □

Algorithm. Our analysis suggests a simple numerical algorithm to solve the planner's problem: start with $b_1 = B_1$ (i.e. pooling) and compute $\overline{S}'(B_1)$ according to equation (17); if $\overline{S}'(B_1) \geq 0$, then $b_1 = B_1$ is the solution, otherwise decrease b_1 until the first-order condition is satisfied or $b_1 = 0$.

The knife-edge case $\overline{S}'(B_1) = 0$ deserves special attention because it pins down the boundary between areas of pooling and partial segmentation. The detailed analysis of this special case is technical and delegated to Appendix B.3. Finally, in Appendix B.4 we fully characterize and illustrate how the meeting technology affects market segmentation for two cases with a geometric meeting technology that is truncated either deterministically or stochastically.

3.3 Comparative Statics

Having solved the planner's problem, we now analyze comparative statics. We are particularly interested in how the optimal allocation of buyers and sellers varies with (i) the

[29] As we show later, the market equilibrium decentralizes the planner's solution and therefore features endogenous market segmentation where both sellers and low-type buyers are indifferent between different segments. Barro and Romer (1987) give a nice example that illustrates how sellers can promise utility by either a low price or fewer other buyers: the Paris metro used to sell expensive first-class tickets for wagons which were physically similar to the second-class ones but which were less crowded in equilibrium.

dispersion in buyer values, and (ii) the properties of the meeting technology. The first comparative static can be thought of as analyzing the effect of an increase in inequality, while the second comparative static can be thought of as analyzing the effect of new technologies like automated resume screening. To simplify exposition, we focus on the example of the geometrically truncated geometric meeting technology, which is characterized by a single parameter σ that reflects the degree of search frictions. In Appendix B.5, we show that the same results hold for general meeting technologies under certain assumptions.

The model now has 4 parameters: x_2, which measures the dispersion of buyer values, σ, which indexes the meeting technology, and B_1 and B_2, the number of low-type and high-type buyers (recall that the number of sellers is normalized to be 1). For a given tuple (x_2, σ, B_1, B_2), Appendix B.4.2 fully characterizes the optimal allocation of buyers and sellers (the planner's solution). Below, we analyze how this allocation changes with σ and x_2.

Changes in Screening Capacity. Since the geometrically truncated geometric meeting technology is indexed by a parameter σ, we represent the meeting technology by $\phi(\mu, \lambda, \sigma)$. The full analytic expression is given by equation (44) in Appendix B.4.2. Since σ measures screening capacity, $\phi(\mu, \lambda, \sigma)$ is increasing in σ. However, the probability that a seller meets at least one buyer, $\phi(\lambda, \lambda, \sigma)$, is independent of σ. That is, a better screening technology increases the probability that a seller finds a high-type buyer but does not change the probability that a seller meets at least one buyer. The following results show that when the screening technology improves, the separation area shrinks and the area where pooling is optimal increases.

Proposition 4. *Given (x_2, σ, B_1, B_2), if the optimal allocation is complete pooling, then for (x_2, σ', B_1, B_2) with $\sigma' > \sigma$, the optimal allocation is again complete pooling.*

Given (x_2, σ, B_1, B_2), if the optimal allocation is complete separation, then for (x_2, σ', B_1, B_2) with $\sigma' < \sigma$, the optimal allocation is again complete separation.

Proof. See the general results of Proposition 11 and 12 in Online Appendix B.5. □

When $\sigma = 0$, meetings are always bilateral and the optimal allocation is always complete separation, since a low-type buyer meeting a seller always crowds out high-type buyers. When $\sigma = 1$, low-type buyers do not impose negative meeting externalities on high type buyers and therefore it is optimal to pool all buyers and sellers in one market. Thus, for any (x_2, B_1, B_2), the above proposition shows that the optimal allocation changes smoothly from complete separation to complete pooling as σ increases from 0 to 1.

Changes in the Dispersion of Buyer Values. Since x_1 is normalized to 1, the parameter x_2 measures the dispersion in buyer values. As we increase x_2, the output loss due to low-type buyers crowding out high-type buyers becomes larger. One may therefore expect that complete separation becomes a more likely outcome while complete pooling becomes less likely. The following proposition presents the formal results.

Proposition 5. *Given (x_2, σ, B_1, B_2), if the optimal allocation is complete pooling, then for (x_2', σ, B_1, B_2) with $x_2' < x_2$, the optimal allocation is again complete pooling.*

Given (x_2, σ, B_1, B_2), if the optimal allocation is complete separation, then for (x_2', σ, B_1, B_2) with $x_2' > x_2$, the optimal allocation is again complete separation.

Proof. This result follows from Proposition 9 and 10 in Online Appendix B.5. □

When $x_2 \to x_1 = 1$ and $\sigma > 0$, then the optimal allocation is complete pooling because the gain from partial separation is negligible, and the planner prefers to pool all buyers and sellers in one place to maximize the matching probability. When x_2 is sufficiently large and $\sigma < 1$, it is optimal to exclude the low-type buyers from participating and set up one market for all sellers and high-type buyers. Thus, for any (σ, B_1, B_2) with $0 < \sigma < 1$, the above proposition shows that the optimal allocation always changes from complete pooling to complete separation as x_2 increases.

As we will show below, when the optimal allocation is complete separation, the optimal allocation can be decentralized by sellers posting fixed prices while when the optimal allocation is complete pooling, then the decentralized equilibrium necessarily involves auctions. Thus as σ changes, the optimal trading mechanism can change accordingly.

4 Market Equilibrium

In this section, we show that no seller can do better in equilibrium than posting a second-price auction combined with either a reserve price or a meeting fee. The reserve price can be positive or negative, where the latter just means that the seller is willing to sell the good at a price below his valuation, which we normalized to 0. Similarly, the meeting fee can be positive, in which case it is paid by each buyer meeting the seller, or negative, in which case payments take place in the opposite direction. Finally, we establish that the equilibrium is constrained efficient.

4.1 Efficiency

Equivalence. To prove constrained efficiency of equilibrium, we show that even if sellers can buy queues directly in a hypothetical competitive market, they cannot do better than in the decentralized environment. In other words, the following two problems are equivalent for sellers.

1. *Sellers' Relaxed Problem*, in which there exists a hypothetical competitive market for queues, with the price for each buyer given by the market utility function. That is, sellers choose a queue (μ, λ) to maximize

$$\pi(\mu, \lambda) = m(\lambda) + (x_2 - 1)\,\phi(\mu, \lambda) - \mu U_2 - (\lambda - \mu)U_1, \tag{19}$$

   where the first two terms are total surplus (7) and the last two terms are the price of the queue.

2. *Sellers' Constrained Problem*, in which sellers must post mechanisms to attract queues of buyers, as described in detail in Section 2. For any mechanism, the corresponding queue must be compatible with the market utility function, which means that it needs to satisfy equation (2). In this case, a seller's profit is again given by equation (19), assuming that sellers post efficient mechanisms, but now queue length and queue composition depend on the posted mechanism.

In the relaxed problem, a seller will "buy" queues of buyers with valuation x_k until their expected marginal contribution T_k to surplus is equal to their marginal cost U_k, where $k = 1, 2$. Hence, if sellers can post a mechanism which delivers buyers their marginal contribution to surplus, then buyers' payoffs are equal to their market utility and the queue is compatible with the mechanism and the market utility function, as defined by equation (2). The following proposition establishes that auctions with an entry fee or a reserve price can achieve this.

Proposition 6. *Any solution (μ, λ) to the sellers' relaxed problem is compatible with an auction with an entry fee in the sellers' constrained problem, where the fee is given by*

$$t = -\frac{(x_2 - 1)\phi_\lambda(\mu, \lambda) + \phi_\lambda(\lambda, \lambda)}{1 - Q_0(\lambda)}. \tag{20}$$

It is also compatible with an auction with a reserve price in the sellers' constrained problem, where the reserve price is given by

$$r = -\frac{(x_2 - 1)\phi_\lambda(\mu, \lambda) + \phi_\lambda(\lambda, \lambda)}{Q_1(\lambda)}. \tag{21}$$

Proof. See Appendix A.6. □

Auctions with Meeting Fees. The intuition behind the case with meeting fees is the following. Recall that a buyer's marginal contribution T_k consists of two parts: (i) a direct effect, representing the fact that the buyer may increase the maximum valuation among the group of buyers meeting the seller,[30] and (ii) an indirect effect, $(x_2 - 1)\phi_\lambda(\mu, \lambda) + \phi_\lambda(\lambda, \lambda)$, representing the externalities that the buyer may impose by making it easier or harder for the seller to meet other buyers. As is well-known, auctions (without reserve prices or fees) provide buyers with a payoff equal to their direct contribution.[31] Buyers' indirect effect on surplus is independent of their type and can therefore be priced by an appropriate entry fee. Since buyers pay the fee whenever they meet a seller, which happens with probability $1 - Q_0(\lambda)$, a meeting fee equal to (20) guarantees that their expected payoff from the mechanism equals exactly T_k, which yields the desired result.

Auctions with Reserve Prices. Perhaps surprisingly, an auction with an appropriate reserve price is also an efficient mechanism that can price all meeting externalities. After all, in contrast to meeting fees, reserve prices may prevent efficient trade. To see this, consider a seller who sets a reserve price $r \in (x_1, x_2)$. Low-type buyers have a zero trading probability at this seller, while their trading probability would be strictly positive at an auction by the same seller with a meeting fee. However, this difference between the two mechanisms only affects out-of-equilibrium behavior; in equilibrium, low-type buyers would visit neither seller. High-type buyers are only affected by the reserve price when they are the only bidder, which happens with probability $Q_1(\lambda)$. A reserve price equal to (21) therefore guarantees that buyers' expected payoff again equals T_k.

Meeting Fees vs. Reserve Prices. Although the meeting fee is a useful instrument from a theoretical point of view, one could argue that it may be difficult to implement in practice. For example, if the meeting fee is positive, fake sellers with no intent to sell could open phantom auctions to collect the meeting fees from interested buyers, which would then discourage buyers from visiting sellers who charge fees in the first place.[32] Those concerns do not apply to auctions with reserve prices. The optimal reserve price has the same sign and plays a similar role as the optimal meeting fee, but is easier to implement because all buyers who do not win, pay (or receive) nothing. If, however, some buyers have valuations below sellers' reservation value and the meeting externalities are positive, then auctions with negative reserve prices are not efficient, while auctions with entry subsidies *and* a reserve price equal to sellers' valuation remain efficient.

Finally, consider Lester et al. (2015), where buyers are ex ante identical and learn their valuation only upon meeting the seller. In their framework, just as in ours, pricing negative

[30] For low-type buyers, the direct effect is given by $\phi_\mu(\lambda, \lambda) = Q_1(\lambda)$, and for high-type buyers, the direct effect is given by $(x_2 - 1)\phi_\mu(\mu, \lambda) + \phi_\mu(\lambda, \lambda)$.

[31] This is easiest to see in a second-price auction. Suppose that the highest and the second highest value are x_2 and x_1. Then, the payoff for the highest value buyer is $x_2 - x_1$, which is also his contribution to surplus. Other bidders receive zero and their contributions to the surplus of the auction are also zero. Extension of this result to other auction formats follows from revenue equivalence.

[32] Similarly, negative meeting fees are subsidies that could attract fake buyers with no intent to purchase.

meeting externalities would require a positive reserve price. However, unlike in our setup, a positive reserve price in their model would actually prevent mutually beneficial trade in equilibrium: as buyers' valuations are only revealed ex post, the highest buyer valuation is between the seller's own valuation and his reserve price with positive probability. This inefficiency prevents sellers from adopting reserve prices in equilibrium, instead they always opt for meeting fees. In Albrecht et al. (2014), who restrict attention to urn-ball meetings ($\phi_\lambda = 0$), this inefficiency does not arise, because sellers always choose to set their reserve price equal to their valuation.

Efficiency. Proposition 6 is an important step towards proving efficiency of the market equilibrium for general meeting technologies, but there is one remaining issue: for a given auction with a reserve price or entry fee, there might be multiple queues compatible with the market utility function. Therefore, even if a solution to the sellers' relaxed problem is compatible with an auction with reserve price or entry fee, it is not clear that sellers will expect that solution to be the realized queue. Most of the literature resolves this issue by assuming that sellers are *optimistic*: a (deviating) seller expects that he can coordinate buyers in such a way that the solution to the sellers' relaxed problem becomes the realized queue.[33] Since this assumption is somewhat arbitrary, we show in the next subsection that we can relax it under some mild restrictions on the meeting technology. However, if we—for the moment—follow the standard approach, then by Proposition 6, a seller's relaxed and constrained problem are equivalent in the sense that they achieve the same outcome. That is, the directed search equilibrium is equivalent to a competitive market equilibrium for queues, which also coincides with the socially efficient planner's allocation.

Proposition 7. *If sellers are optimistic, the directed search equilibrium is constrained efficient for any meeting technology.*

Proof. See Appendix A.7. □

Hence, we have shown that despite the potential presence of spillovers in the meeting process, business stealing externalities and agency costs, the competing mechanisms problem reduces to one where sellers can buy queues in a competitive market. This result, of course, requires a sufficiently large contract space. If it is not possible for sellers to either commit to a reserve price above their valuation or charge fees, the decentralized equilibrium will only be efficient for invariant meeting technologies (i.e. $\phi_\lambda = 0$). If $\phi_\lambda < 0$ (resp. > 0), buyers impose negative (resp. positive) externalities on other meetings and will receive more (resp. less) than their marginal social contribution.[34]

[33] See, for example, Eeckhout and Kircher (2010a,b).

[34] With free entry of sellers, the buyer-seller ratio would be too high (resp. too low) in this case.

Equivalence of Different Mechanisms. As shown before, the planner's solution is unique; however, the equilibrium mechanism is not unique: it can be decentralized in multiple ways. Note that in the submarket with low-type buyers only, a second-price auction with a reserve price is equivalent to price posting. So price posting can also be an equilibrium mechanism.

4.2 Uniqueness of Beliefs

The efficiency result in proposition 7 assumed that sellers are optimistic. Without this assumption, it is not clear how sellers should evaluate the expected payoff of deviations if multiple queues are compatible with market utility. In this subsection, we show that such a scenario is rather special in the sense that—under mild restrictions on the meeting technology—the solution to the market utility condition is in fact unique, rendering the optimism assumption redundant.

Uniqueness. The following proposition then presents our result regarding uniqueness of the beliefs for a seller posting a second-price auction with a reserve price. To avoid the situation where the optimal response of low-type buyers is indeterminate, we assume that their market utility is strictly positive.

Proposition 8. *Assume that $U_1 > 0$. Under assumptions 1 and 2, for each seller posting a second-price auction with a reserve price r, there is a unique queue (μ, λ) compatible with the market utility function. Furthermore, for two sellers posting reserve prices r^a and r^b, it holds that $\lambda^a > \lambda^b$ if and only if $r^a < r^b$.*

Proof. See Appendix A.8. □

If both sellers attract low-type buyers, then the expected payoffs for low-type buyers from visiting any of the two sellers must be the same: $Q_1(\lambda^a)(1 - r^a) = Q_1(\lambda^b)(1 - r^b)$, which implies that $\lambda^a > \lambda^b$ if and only if $r^a < r^b$, since $Q_1(\lambda)$ is strictly decreasing by Assumption 2. When one seller attracts low-type buyers and the other does not, the latter seller must have posted a high reserve price implying a shorter queue without low-type buyers.

Things are slightly more complicated when sellers post a second-price auction with an entry fee. Below, we introduce one weak additional restriction on the meeting technology, which is sufficient to guarantee that there exists a monotonic relation between meeting fees and queue lengths. This implies that there exists a unique queue that is compatible with the market utility function when sellers post an auction with an entry fee.

Assumption 4. $(1 - Q_0(\lambda))/Q_1(\lambda)$ *is weakly increasing in* λ.

If we rewrite $(1-Q_0(\lambda))/Q_1(\lambda)$ as $1+\sum_{k=2}^{\infty} Q_k(\lambda)/Q_1(\lambda)$, then this assumption states that with a higher buyer-seller ratio, it is relatively more likely that a buyer will meet competitors in an auction rather than being alone.

Proposition 9. *Under assumptions 1, 2, and 4, for each seller posting an auction with entry fee t, there is a unique queue (μ, λ) compatible with the market utility function. Furthermore, for two sellers posting entry fees t^a and t^b, it holds that $\lambda^a > \lambda^b$ if and only if $t^a < t^b$.*

Proof. See Appendix A.9. □

The intuition behind Proposition 9 is similar to that of Proposition 8 and readily follows from the correspondence between the reserve price and entry fee: $t = rQ_1/(1 - Q_0)$. Again, consider two different queues a and b. We have shown in Proposition 8 that $\lambda^a > \lambda^b$ if and only if $r^a < r^b$. Under Assumption 4, the two inequalities jointly lead to $t^a < t^b$.

Hence, we have established that under mild restrictions on the meeting technology, there exists only one queue which is compatible with market utility when sellers post an auction with a reserve price or an entry fee. Consequently, the assumption that sellers are optimistic is redundant for a large class of meeting technologies.

5 N Buyer Types

5.1 Surplus

In this section, we consider the case with N buyer values: $0 < x_1 < \cdots < x_N$. The measure of x_k buyers is B_k for $k = 1, \ldots, N$. The rest of the model remains the same, including the planner's problem and the definition of the decentralized equilibrium. For example, we can continue to denote the queue of a submarket by $\boldsymbol{q} = \{q_1, q_2, \ldots, q_N\}$ where q_k is the number of x_k buyers per seller. To use our alternative representation of meeting technologies, we apply a change of notation and define μ_k as the queue length of buyers with value x_k or higher, i.e. $\mu_k = q_k + \cdots + q_N$ for $k = 1, \ldots, N$. The queue in the submarket can then be represented by $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_N)$, where μ_1 is the total queue length. Thus $\phi(\mu_k, \mu_1)$ is the probability that a seller meets at least one buyer with value x_k or higher. We further adopt the convention $x_0 \equiv 0$ and $\mu_{N+1} \equiv 0$ to simplify notation.

The following Lemma extends Lemma 1 and 2 to the case of N buyer values. It turns out that this general case does not add much complexity. The interpretation of equations (22) to (25) closely resembles the corresponding interpretation in the two-type case. Here, we only discuss equation (23) as an example and omit the others.[35] In equation (23), the first term of $T_k(\boldsymbol{\mu})$ reflects the direct contribution to surplus of a buyer with valuation x_k when this buyer has the highest value in an n-to-1 meeting; this contribution equals the difference between

[35] A similar result appears in Cai et al. (2017), so it is worth emphasizing that the credit belongs with the current paper: as they explicitly acknowledge in their article, Cai et al. (2017) borrow Lemma 1 directly from our paper, of which a first draft was written in 2016. The same applies to a number of other results, e.g. Proposition 6 here vs. Proposition 4 in Cai et al. (2017).

the highest and the second-highest buyer values. The second term of $T_k(\boldsymbol{\mu})$ represents the externalities that the buyer may impose on other buyers and the seller. It does not depend on k, because the meeting function treats all buyers symmetrically. Specifically, if a buyer makes it easier for the other buyers to meet the seller ($\phi_\lambda \geq 0$), he increases total surplus through a positive meeting externality, even if he does not have the highest value . A similar logic applies for negative meeting externalities ($\phi_\lambda \leq 0$).

Lemma 3. *Consider a submarket with a measure 1 of sellers and a queue* $\boldsymbol{\mu} \equiv (\mu_1, \dots, \mu_N)$ *of buyers. Total surplus in the submarket then equals*

$$S(\boldsymbol{\mu}) = \sum_{j=1}^{N} (x_j - x_{j-1}) \phi(\mu_j, \mu_1) \tag{22}$$

The marginal contribution to surplus of a buyer with valuation x_k *equals*

$$T_k(\boldsymbol{\mu}) = \sum_{j=1}^{k} (x_j - x_{j-1}) \phi_\mu(\mu_j, \mu_1) + \sum_{j=1}^{N} (x_j - x_{j-1}) \phi_\lambda(\mu_j, \mu_1). \tag{23}$$

A seller's marginal contribution to surplus equals

$$R(\boldsymbol{\mu}) = \sum_{j=1}^{N} (x_j - x_{j-1}) \left[\phi(\mu_j, \mu_1) - \mu_j \phi_\mu(\mu_j, \mu_1) - \mu_1 \phi_\lambda(\mu_j, \mu_1)\right]. \tag{24}$$

The Hessian matrix of the surplus function $S(\boldsymbol{\mu})$ *in equation* (22) *is negative definite if and only if*

$$-m''(\mu_1)x_1 - \sum_{k=2}^{N} (x_k - x_{k-1}) \left(\phi_{\lambda\lambda}(\mu_k, \mu_1) - \frac{\phi_{\mu\lambda}(\mu_k, \mu_1)^2}{\phi_{\mu\mu}(\mu_k, \mu_1)} \right) > 0 \tag{25}$$

Proof. See Appendix B.1. □

Decentralized Equilibrium. We can again show that in the decentralized equilibrium, sellers can do no better than posting second-price auctions with an entry fee or a reserve price, and their relaxed problem and constrained problem are equivalent so that the decentralized equilibrium is constrained efficient. Also the uniqueness result which makes the assumption on optimistic beliefs redundant continues to hold. For the proofs and derivations, we refer to the Online Appendix.

5.2 Queues Across Submarkets

A larger number of buyer types increases the complexity of the planner's problem. Although the result in Proposition 2 generalizes in a straightforward way—i.e., with N types of buyers, no more than $N+1$ submarkets are required—a full characterization of these submarkets quickly becomes intractable.[36] Nevertheless, we provide a partial characterization by showing that we can compare queue compositions between any two submarkets in terms of first-order stochastic dominance, under Assumptions 1 and 2.

First-Order Stochastic Dominance. Consider two arbitrary submarkets, indexed by $\ell \in \{a, b\}$, that attract a queue $\boldsymbol{\mu}^\ell$ of buyers. The following proposition compares queue compositions between the submarkets in terms of first-order stochastic dominance.

Proposition 10. *Consider two submarkets a and b with respective queues $\boldsymbol{\mu}^a$ and $\boldsymbol{\mu}^b$, satisfying $\mu_1^a > \mu_1^b$. If assumptions 1 and 2 hold, then for any k,*

$$\frac{\mu_k^b}{\mu_1^b} \geq \frac{\mu_k^a}{\mu_1^a}. \tag{26}$$

Proof. See Appendix B.2. □

This result is quite remarkable. It shows that under two weak assumptions on the meeting technology, the buyer value distribution of a short queue always first-order stochastically dominates that of a long queue. A simple consequence of this result is that the shorter queue always has a weakly higher upper and a weakly higher lower bound.

To understand the above proposition, assume that buyers with valuations x_{k-1} and x_k both visit submarkets a and b with positive probability. Since at the planner's solution, the marginal contribution of buyers with valuations x_{k-1} and x_k must be the same across the two submarkets, by equation (23) we have $\phi_\mu(\mu_k^a, \mu_1^a) = \phi_\mu(\mu_k^b, \mu_1^b)$. If $\mu_1^a > \mu_1^b$, by assumption 2, queue a must have a lower proportion of buyers with values weakly greater than x_k.

Proposition 10 offers some testable implications that do not require characterization of the entire model. Also, since the assumptions are rather weak, they apply to almost all meeting technologies that are currently used in the literature. Consider for example two identical goods that are offered on eBay where the queue lengths and the buyer value distributions

[36] Unless, of course, the meeting technology is bilateral or jointly concave, which lead to perfect separation and perfect pooling, respectively. We conjecture that the following result, which is similar to Proposition 3, continues to hold: At the social optimum, there will be one submarket for all x_N buyers. Note that this conjecture has sharp predictions. If we take the submarket for x_N buyers out, then by the same logic, in the remaining submarkets there will be exactly one which contains all the remaining x_{N-1} buyers. Repeating this logic, implies then that there will be N submarkets where the highest buyer type in the ℓ-th submarket is $x_{N+1-\ell}$ and some submarkets can be idle. However, we were unable to prove this conjecture.

differ. Our theory derives a sharp prediction on the relation between the queue length and the buyer value distribution.

Proposition 10 is useful beyond the specific environment that we consider here. To see this, suppose that we add an epsilon degree of seller heterogeneity to the model. The equilibrium allocation of buyers and sellers will then change marginally. Without Proposition 10, we cannot order the resulting buyer value distributions of different types of sellers, making an analysis of sorting in terms of first-order stochastic dominance impossible. In other words, Proposition 10 forms the foundation for the theory of sorting with multilateral meetings that we develop in Cai et al. (2018).

6 Conclusion

In this paper, we analyze a directed search model where sellers compete for heterogeneous buyers by posting trading mechanisms. We have shown how the meeting process between buyers and sellers affects the equilibrium selling mechanisms and market segmentation. This framework can help us to understand why sellers who use online meeting tools often also use auctions as selling mechanism. Concerning market segmentation, when low-valuation buyers reduce the probability that sellers and high-valuation buyers meet, sellers will discourage the low-valuation buyers from visiting. This can lead to complete or partial market segmentation, depending on the dispersion of valuations and the degree of congestion in the meeting process. All high-valuation buyers are always in one segment, either with or without a subset of low value buyers.

We also introduce a new function ϕ which makes the analysis of general meeting technologies tractable and allows us to generalize the competing mechanism literature. Using this function, we show that in a large economy, despite the presence of private information and possible search externalities, the directed search equilibrium is equivalent to a competitive equilibrium (where the commodities are buyer types and the prices are the market utilities). A seller can attract a desired queue by posting an auction with entry fee or by charging an appropriate reserve price. Finally, we introduced conditions on the meeting technology such that for any given market utility function, the queue attracted by an auction with reserve price or entry fee is unique. This is necessary to establish the equivalence between the two equilibria.

Appendix A Proofs

A.1 Proof of Proposition 1

When $n = 0$, equation (5) is simply $P_0(\lambda) = 1 - \phi(\lambda, \lambda)$. When $n \geq 1$, by equation (4),

$$\frac{\partial^n}{\partial \mu^n} (1 - \phi(\mu, \lambda)) = \sum_{k=n}^{\infty} P_k(\lambda) n! \left(-\frac{1}{\lambda}\right)^n \left(1 - \frac{\mu}{\lambda}\right)^{k-n}.$$

Evaluating the above equation at $\mu = \lambda$ yields equation (5). □

A.2 Proof of Proposition 2

Recall that the social planner's problem is given by (11). Below, we rewrite (11) slightly by introducing a new function $\widehat{S}$, total surplus per agent, which has two advantages: i) the domain of $\widehat{S}$ is compact, and ii) the accounting constraints for buyers and sellers hold with equalities so that we can apply directly the Fenchel-Bunt Theorem.

Suppose that the planner creates $\widetilde{L}$ submarkets, which may include an inactive one. In submarket ℓ, the measure of sellers is $\widetilde{\alpha}^\ell$ and the measure of buyers with value x_j is $\widetilde{B}_j^\ell$ for $j = 1, 2$. Therefore, $\sum_{\ell=1}^{\widetilde{L}} \widetilde{\alpha}^\ell = 1$ and $\sum_{\ell=1}^{\widetilde{l}} \widetilde{B}_j^\ell = B_j$ for $j = 1, 2$. Define $\widetilde{z}_1^\ell = (\widetilde{B}_1^\ell + \widetilde{B}_2^\ell)/(\alpha^\ell + \widetilde{B}_1^\ell + \widetilde{B}_2^\ell)$ and $\widetilde{z}_2^\ell = \widetilde{B}_2^\ell/(\alpha^\ell + \widetilde{B}_1^\ell + \widetilde{B}_2^\ell)$, i.e. $\widetilde{z}_1^\ell$ is the fraction of buyers and $\widetilde{z}_2^\ell$ is the fraction of x_2 buyers in a submarket ℓ.

Since total surplus in each submarket exhibits constant returns to scale with respect to the number of sellers and the number of high-type and low-type buyers, we can normalize the total number of buyers and sellers in each submarket (active or inactive) to 1, and define the surplus per agent (both buyers and sellers) in submarket ℓ as $\widehat{S}(\widetilde{z}_1^\ell, \widetilde{z}_2^\ell)$. When $\widetilde{\alpha}^\ell > 0$, it is given by

$$\widehat{S}(z_1, z_2) = \frac{1}{\alpha^\ell + \widetilde{B}_1^\ell + \widetilde{B}_2^\ell} \cdot \widetilde{\alpha}^\ell S\left(\frac{\widetilde{B}_2^\ell}{\widetilde{\alpha}^\ell}, \frac{\widetilde{B}_2^\ell + \widetilde{B}_1^\ell}{\widetilde{\alpha}^\ell}\right).$$

and when $\widetilde{\alpha}^\ell = 0$, it is simply zero. The function $\widehat{S}$ is well defined even in a submarket with buyers only ($\widetilde{z}_1^\ell = 1$) and its domain is compact. The total surplus generated from all submarkets is

$$\sum_{\ell=1}^{\widetilde{L}} (\alpha^\ell + \widetilde{B}_1^\ell + \widetilde{B}_2^\ell) \widehat{S}(\widetilde{z}_1^\ell, \widetilde{z}_2^\ell).$$

Therefore, as in equation (11), total surplus is a convex combination of the individual submarkets' surpluses, which are represented by $\widehat{S}$ here. The planner's solution is thus the supreme of all such convex combinations. Because the function $\widehat{S}$ is continuous and its domain is compact, the graph of $\widehat{S}$ is compact, which implies that the convex hull of the graph is also compact (see, for example, Theorem 17.2 of Rockafellar (1970)). Thus the supreme can be reached as a maximum. Furthermore, note that the domain of $\widehat{S}$ is the set $\{(z_1, z_2) \,|\, 0 \leq z_2 \leq z_1 \leq 1\}$, which is connected. By the Fenchel-Bunt Theorem [see Theorem 18 (ii) of Eggleston (1958)], which is an extension of Caratheodory's theorem, it suffices to create 3 submarkets. □

A.3 Proof of Lemma 2

This lemma is a special case of Lemma 3. Nevertheless, we give a short proof here. The Hessian matrix of the surplus function is

$$\begin{pmatrix} S_{\mu\mu}(\mu,\lambda) & S_{\mu\lambda}(\mu,\lambda) \\ S_{\mu\lambda}(\mu,\lambda), & S_{\lambda\lambda}(\mu,\lambda) \end{pmatrix} = \begin{pmatrix} (x_2-x_1)\phi_{\mu\mu}(\mu,\lambda), & (x_2-x_1)\phi_{\mu\lambda}(\mu,\lambda) \\ (x_2-x_1)\phi_{\mu\lambda}(\mu,\lambda), & x_1 m''(\lambda) + (x_2-x_1)\phi_{\lambda\lambda}(\mu,\lambda) \end{pmatrix}$$

Since $\phi_{\mu\mu}$ is always negative, by Sylvester's criterion the Hessian matrix is negative semidefinite if and only if its determinant is positive. That is,

$$(x_2-x_1)\phi_{\mu\mu}(\mu,\lambda)\left(x_1 m''(\lambda) + (x_2-x_1)\phi_{\lambda\lambda}(\mu,\lambda)\right) - (x_2-x_1)^2\phi_{\mu\lambda}(\mu,\lambda)^2 > 0$$

Dividing both sides by $(x_2-x_1)\phi_{\mu\mu}(\mu,\lambda)$ gives (12). □

A.4 Collection of Technical Lemmas

Below, we collect several technical lemmas which will be useful to characterize the planner's solution established in Proposition 3. The first lemma addresses the participation problem.

Lemma 4. *The planner will assign ...*

i) all high-type buyers to active submarkets under assumption 1.

ii) all low-type buyers to active submarkets if $\phi_\lambda(\lambda z,\lambda) \geq 0$ for all z and λ.

iii) all sellers to active submarkets under assumption 1.

Proof. Part i) is explained in he text after Lemma 4. For part ii), assume $\phi_\lambda(\mu,\lambda) \geq 0$. By equation (8) we have $T_1 > 0$. Hence, buyers' marginal contribution to surplus is always positive in this case.

For iii), since $\phi(\lambda z,\lambda)$ is strictly concave in λ for $z > 0$, we have $\phi(\lambda z,\lambda) > \lambda\frac{\partial\phi(\lambda z,\lambda)}{\partial\lambda}$. For $z = 1$, this condition reduces to $m(\lambda) > \lambda m'(\lambda)$. Also note that $\lambda\frac{\partial\phi(\lambda z,\lambda)}{\partial\lambda} = \mu\phi_\mu(\mu,\lambda) + \lambda\phi_\lambda(\mu,\lambda)$. Thus $R > 0$ in equation (10). □

An alternative way of understanding Assumption 1 is the following. Holding fixed the number of low-type and high-type buyers in a submarket, adding one more seller decreases the queue length but keeps the fraction of high-type buyers constant. Assumption 1 then implies that the total surplus in this submarket is always concave in the number of sellers.

Lemma 5. *Consider a submarket where the measure of sellers, low-type buyers and high-type buyers are α, b_1, and b_2 respectively. Under Assumption 1, total surplus $\alpha S\left(\frac{b_2}{\alpha}, \frac{b_1+b_2}{\alpha}\right)$ is strictly concave in α.*

Proof. Surplus generated from the submarket is $\alpha S(\frac{b_2}{\alpha}, \frac{b_1+b_2}{\alpha})$. Define $\widetilde{b} = b_1+b_2$ and $z = b_2/\widetilde{b}$. Then the second-order derivative of the surplus function, $\alpha S(\frac{b_2}{\alpha}, \frac{b_1+b_2}{\alpha})$, with respect to α is

$$\frac{\widetilde{b}^2}{\alpha^3}\left[m''(\frac{\widetilde{b}}{\alpha}) + (x_2-1)\left(z^2\phi_{\mu\mu}\left(\frac{z\widetilde{b}}{\alpha}, \frac{\widetilde{b}}{\alpha}\right) + 2z\phi_{\mu\lambda}\left(\frac{z\widetilde{b}}{\alpha}, \frac{\widetilde{b}}{\alpha}\right) + \phi_{\lambda\lambda}\left(\frac{z\widetilde{b}}{\alpha}, \frac{\widetilde{b}}{\alpha}\right)\right)\right]$$

Note that by Assumption 1, $m''(\frac{\widetilde{b}}{\alpha})$ is strictly negative and the second term in the bracket is $\frac{\partial^2\phi(\lambda z,\lambda)}{\partial\lambda^2}$ with $\lambda = \widetilde{b}/\alpha$ and hence is weakly negative, which implies that the second-order derivative with respect to α is strictly negative. □

The next lemma formally shows that $R(\lambda z, \lambda)$, sellers' marginal contribution to surplus, is increasing in both z and λ. Hence the level curve of $R(\lambda z, \lambda)$ is downward-sloping.

Lemma 6. *Under Assumptions 1 and 2, sellers' marginal contribution $R(\lambda z, \lambda)$ is strictly positive, and strictly increasing in z and in λ. The level curves of $R(\lambda z, \lambda)$ are therefore downward sloping in the λ-z plane. Under the same assumptions, high-type buyers' marginal contribution $T_2(\lambda z, \lambda)$ is strictly positive, and strictly decreasing in z and in λ.*

Proof. Lemma 4 already showed that $R(\lambda z, \lambda)$ is always strictly positive. The discussions before Lemma 6 showed that $\partial R(\lambda z, \lambda)/\partial z > 0$ and $\partial R(\lambda z, \lambda)/\partial \lambda > 0$ so we only need to consider $T_2(\lambda z, \lambda)$.

First note that by equation (9), $\partial T_2(\lambda z, \lambda)/\partial z = (x_2-x_1)\lambda(\phi_{\mu\mu}(\lambda z, \lambda)+\phi_{\mu\lambda}(\lambda z, \lambda))$. Note that $z\phi_{\mu\mu}+\phi_{\mu\lambda} < 0$ by Assumption 2 and $(1-z)\phi_{\mu\mu} \leq 0$. Therefore, $\phi_{\mu\mu}(\lambda z, \lambda)+\phi_{\mu\lambda}(\lambda z, \lambda) < 0$, which implies that $T_2(\lambda z, \lambda)$ is strictly decreasing in z. Furthermore, $T_2(\lambda z, \lambda) \geq T_2(\lambda, \lambda) = x_2 m'(\lambda) > 0$.

Next, we have $\partial T_2(\lambda z, \lambda)/\partial \lambda = x_1 m''(\lambda) + (x_2-x_1)(z\phi_{\mu\mu}(\lambda z, \lambda) + (1+z)\phi_{\mu\lambda}(\lambda z, \lambda) + \phi_{\lambda\lambda}(\lambda z, \lambda))$. Finally, note that,

$$z\phi_{\mu\mu} + (1+z)\phi_{\mu\lambda} + \phi_{\lambda\lambda} = (1-z)\left(z\phi_{\mu\mu} + \phi_{\mu\lambda}\right) + \left(z^2\phi_{\mu\mu} + 2z\phi_{\mu\lambda} + \phi_{\lambda\lambda}\right).$$

The first term on the right-hand side is negative by Assumption 2, and the second term is negative by Assumption 1. Furthermore, the first term is zero if and only if $z = 1$, in which case the second term is strictly negative. Therefore, $\partial T_2(\lambda z, \lambda)/\partial \lambda < 0$. □

Our next lemma shows how $T_1(\lambda z, \lambda)$ and $T_2(\lambda z, \lambda)$, (the marginal contribution to surplus by a buyer) vary along a level curve of $R(\lambda z, \lambda)$ and how this depends on the sign of the determinant of the Hessian matrix of the surplus function.

Lemma 7. *For any given R^*, let $\lambda(z)$ be implicitly determined by the level curve $R(\lambda z, \lambda) = R^*$. Then, $T_2(\lambda(z)z, \lambda(z))$ is strictly decreasing in z if the determinant of the Hessian of the surplus function in equation (7) is strictly positive (and decreasing if the determinant is strictly negative). The reverse result holds for $T_1(\lambda(z)z, \lambda(z))$.*

Proof. Since $T_1(\lambda z, \lambda)$ and $R(\lambda z, \lambda)$ are given by Equation (8) and (10), respectively, we have

$$\frac{dT_1(\lambda z, \lambda)}{dz}\Big|_{R(\lambda z,\lambda)=R^*} = \frac{\partial T_1(\lambda z, \lambda)}{\partial z} + \frac{\partial T_1(\lambda z, \lambda)}{\partial \lambda}\left(-\frac{\partial R(\lambda z, \lambda)/\partial z}{\partial R(\lambda z, \lambda)/\partial \lambda}\right)$$
$$= (x_2-1)\lambda\phi_{\mu\lambda}+(m''(\lambda) + (x_2 - 1)(z\phi_{\mu\lambda} + \phi_{\lambda\lambda}))\left(-\frac{-(x_2 - 1)\lambda(z\phi_{\mu\mu} + \phi_{\mu\lambda})}{-\lambda m''(\lambda) - (x_2 - 1)\lambda(z^2\phi_{\mu\mu} + 2z\phi_{\mu\lambda} + \phi_{\lambda\lambda})}\right).$$

where in the second line we have suppressed the arguments $(\lambda z, \lambda)$ from the relevant functions. Simplifying the above equation gives

$$\frac{dT_1(\lambda z, \lambda)}{dz}\Big|_{R(\lambda z,\lambda)=R^*} = \frac{z(x_2 - 1)\lambda\phi_{\mu\mu}}{m''(\lambda) + (x_2 - 1)\left(z^2\phi_{\mu\mu} + 2z\phi_{\mu\lambda} + \phi_{\lambda\lambda}\right)}\left(-m''(\lambda) - (x_2 - 1)(\phi_{\lambda\lambda} - \frac{\phi^2_{\mu\lambda}}{\phi_{\mu\mu}})\right)$$

To prove the results regarding $T_2(\lambda z, \lambda)$, we first consider how $\phi_\mu(\lambda z, \lambda)$ varies along a level curve of $R(\lambda z, \lambda)$.

$$\frac{d\phi_\mu(\lambda z, \lambda)}{dz}\Big|_{R(\lambda z,\lambda)=R^*} = \frac{\partial \phi_\mu(\lambda z, \lambda)}{\partial z} + \frac{\partial \phi_\mu(\lambda z, \lambda)}{\partial \lambda}\left(-\frac{\partial R(\lambda z, \lambda)/\partial z}{\partial R(\lambda z, \lambda)/\partial \lambda}\right)$$
$$= \lambda\phi_{\mu\mu} + (z\phi_{\mu\mu} + \phi_{\mu\lambda})\left(-\frac{-(x_2 - 1)\lambda(z\phi_{\mu\mu} + \phi_{\mu\lambda})}{-\lambda m''(\lambda) - (x_2 - 1)\lambda(z^2\phi_{\mu\mu} + 2z\phi_{\mu\lambda} + \phi_{\lambda\lambda})}\right),$$

which can be simplified to

$$\frac{1}{z(x_2 - 1)}\left(-\frac{z(x_2 - 1)\lambda\phi_{\mu\mu}}{m''(\lambda) + (x_2 - 1)\left(z^2\phi_{\mu\mu} + 2z\phi_{\mu\lambda} + \phi_{\lambda\lambda}\right)}\left(m''(\lambda) + (x_2 - 1)(\phi_{\lambda\lambda} - \frac{\phi^2_{\mu\lambda}}{\phi_{\mu\mu}})\right)\right)$$

Note that the above equation is simply $\frac{1}{z(x_2-1)}\frac{dT_1(\lambda z,\lambda)}{dz}\big|_{R(\lambda z,\lambda)=R^*}$. Since $T_2(\lambda z, \lambda) = T_1(\lambda z, \lambda)+$

$(x_2 - 1)\phi_\mu(\lambda z, \lambda)$, we have thus proved that

$$\frac{d\left(T_2(\lambda z, \lambda) - T_1(\lambda z, \lambda)\right)}{dz}\Big|_{R(\lambda z,\lambda)=R^*} = -\frac{1}{z}\frac{dT_1(\lambda z, \lambda)}{dz}\Big|_{R(\lambda z,\lambda)=R^*}$$

which is equal to

$$\frac{dT_2(\lambda z, \lambda)}{dz}\Big|_{R(\lambda z,\lambda)=R^*} = -\frac{1-z}{z}\frac{dT_1(\lambda z, \lambda)}{dz}\Big|_{R(\lambda z,\lambda)=R^*} \tag{27}$$

□

Our last lemma shows the single-crossing result, which complements Lemma 7 and is critical to the result that at the social optimum, there is one submarket for all x_2 buyers.

Lemma 8. *Under Assumption 3, each level curve of $R(\lambda z, \lambda)$ intersects with the curve $H(\lambda z, \lambda) = 1/(x_2 - 1)$ at most once and from the left in the z-λ plane.*

Proof. By direct computation we have,

$$\frac{\partial R(\lambda z, \lambda)/\partial z}{\partial R(\lambda z, \lambda)/\partial \lambda}\Big|_{x_2=1+1/H(\lambda z,\lambda)} = \frac{\lambda\phi_{\mu\mu}(\lambda z, \lambda)}{z\phi_{\mu\mu}(\lambda z, \lambda) + \phi_{\mu\lambda}(\lambda z, \lambda)} = \frac{\partial\phi_\mu(\lambda z, \lambda)/\partial z}{\partial\phi_\mu(\lambda z, \lambda)/\partial \lambda}.$$

Suppose that a level curve of $R(\lambda z, \lambda)$ intersects with the curve $H(\lambda z, \lambda) = 1/(x_2 - 1)$ at point $(\lambda z, \lambda)$ (with a slight abuse of notation). Then, x_2 is given by $1 + 1/H(\lambda z, \lambda)$ and the left-hand side of the above equation denotes the slope of the level curve of $R(\lambda z, \lambda)$ at point $(\lambda z, \lambda)$ in the z-λ plane. Assumption 3 then implies Lemma 8. □

A.5 Proof of Proposition 3

Before moving to the main part of the proof, we need the following simple mathematical fact. Suppose that $f(x, y)$ is an arbitrary function and is strictly concave in y. Furthermore, y as a function of x is implicitly defined by $f_2(x, y(x)) = 0$ (subscripts of f indicate partial derivatives).

Differentiating with respect to x gives $y'(x) = -f_{12}(x, y(x))/f_{22}(x, y(x))$. Next, $\frac{d}{dx}f(x, y(x)) = f_1(x, y(x))$, and

$$\frac{d^2}{dx^2}f(x, y(x)) = f_{11}(x, y(x)) - \frac{f_{11}(x, y(x))^2}{f_{22}(x, y(x))} \tag{28}$$

Therefore, $f(x, y(x))$ is locally concave in x if and only if $f(x, y)$ is locally concave in (x, y).

After the above preparation, we now proceed to the main proof. As mentioned before Proposition 3, we need to prove the claim that if $\overline{S}'(b_1) = 0$, then $\overline{S}''(b_1) < 0$. Recall that the two-step problem of the planner is

$$\max_{b_1} \max_{\alpha} \quad \widetilde{S}(b_1,\alpha) \equiv \alpha S(\frac{B_2}{\alpha}, \frac{B_2+b_1}{\alpha}) + (1-\alpha)S(0, \frac{B_1-b_1}{1-\alpha}). \tag{29}$$

Thus $\overline{S}(b_1) = \max_\alpha \widetilde{S}(b_1,\alpha)$. Note that by Lemma 5, $\widetilde{S}(b_1,\alpha)$ is always strictly concave in α. We define the first term on the right-hand side as $\widetilde{S}^a(b_1,\alpha)$ and the second term as $\widetilde{S}^b(b_1,\alpha)$.

As we mentioned in the main text, the planner's solution can be solved by at most two submarkets, one of which contains all high-type buyers. We first consider the case in which the marginal contribution to surplus of x_1 buyers in the first submarket is strictly positive, i.e., $T_1(\frac{B_2}{\alpha^*(b_1)}, \frac{B_2+b_1}{\alpha^*(b_1)}) > 0$. In this case, T_1 and R are the same between the two submarkets, and the optimal $\alpha^*(b_1)$ is characterized by the first-order condition $\widetilde{S}_2(b_1,\alpha) = 0$ (subscripts of S, $\widetilde{S}$, $\widetilde{S}^a$, and $\widetilde{S}^b$ indicate partial derivatives), or equivalently $R(\frac{B_2}{\alpha^*}, \frac{B_2+b_1}{\alpha^*}) = R(0, \frac{B_1-b_1}{1-\alpha^*})$ as in equation (15). In this case, by equation (28) to show that $\overline{S}''(b_1) < 0$ we only need to show that $\widetilde{S}(b_1,\alpha)$ is locally concave in (b_1,α). In the following we will show both $\widetilde{S}^a(b_1,\alpha)$ and $\widetilde{S}^b(b_1,\alpha)$ are locally concave in (b_1,α). By Lemma 5, both $\widetilde{S}^a(b_1,\alpha)$ and $\widetilde{S}^b(b_1,\alpha)$ are strictly concave in α.

Consider $\widetilde{S}^b(b_1,\alpha)$ first. Since $\widetilde{S}^b(b_1,\alpha) = (1-\alpha)S(0, \frac{B_1-b_1}{1-\alpha}) = (1-\alpha)m(\frac{B_1-b_1}{1-\alpha})$, we have

$$\widetilde{S}^b_{11}(b_1,\alpha)\widetilde{S}^b_{22}(b_1,\alpha) - \widetilde{S}^b_{12}(b_1,\alpha)^2 = 0$$

Thus $\widetilde{S}^b(b_1,\alpha)$ is always concave in (b_1,α). Next, consider $\widetilde{S}^a(b_1,\alpha)$. Since both R and T_1 are the same between the two submarkets, by Assumption 3 and Lemma 7, $S(\mu,\lambda)$ must be locally concave at point $(\frac{B_2}{\alpha}, \frac{B_2+b_1}{\alpha})$. Furthermore, note that

$$\widetilde{S}^a_{11}(b_1,\alpha)\widetilde{S}^a_{22}(b_1,\alpha) - \widetilde{S}^a_{12}(b_1,\alpha)^2 = \frac{B_2^2}{\alpha^4}\left(S_{11}(\frac{B_2}{\alpha}, \frac{B_2+b_1}{\alpha})S_{22}(\frac{B_2}{\alpha}, \frac{B_2+b_1}{\alpha}) - S_{12}(\frac{B_2}{\alpha}, \frac{B_2+b_1}{\alpha})^2\right)$$

Thus $\widetilde{S}^a(b_1,\alpha)$ is locally concave in (b_1,α).

Next we consider the case $T_1(\frac{B_2}{\alpha^*(b_1)}, \frac{B_2+b_1}{\alpha^*(b_1)}) = 0$, which then implies $R(\frac{B_2}{\alpha^*(b_1)}, \frac{B_2+b_1}{\alpha^*(b_1)}) \geq 1$ and $\alpha^*(b_1) = 1$. Therefore, $\widetilde{S}^b(b_1,\alpha)$ is zero, and we only need to show that the first term, $\widetilde{S}^a(b_1,\alpha)$, is locally concave at point $(b_1, 1)$. To proceed, we need the following lemma.

Lemma 9. *Under Assumption 1, 2, and 3, if at some point (μ_0,λ_0), $R(\mu_0,\lambda_0) \geq 1$ and $T_1(\mu_0,\lambda_0) = 0$, then the Hessian matrix of $S(\mu,\lambda)$ at point (μ_0,λ_0) is negative definitive.*

Proof. Step 1: For any given z, $\lim_{\lambda\to\infty} T_1(\lambda z,\lambda) = 0$. To see this, note that $\lim_{\lambda\to\infty} m'(\lambda) = 0$, by equation 8 we only need to show that $\lim_{\lambda\to\infty} \phi_\lambda(\lambda z,\lambda) = 0$. Because $\phi(\mu,\lambda)$ is always concave in μ, we have $\phi(\lambda z,\lambda) > \lambda z \phi_\mu(\lambda z,\lambda)$. For $z > 0$, this implies that $\lim_{\lambda\to\infty} \phi_\mu(\lambda z,\lambda) \leq \lim_{\lambda\to\infty} \phi(\lambda z,\lambda)/\lambda z = 0$. Note that $\lim_{\lambda\to\infty} \phi_\mu(0,\lambda) = 0$ simply by continuity. Next, Assumption 1 implies that $\lim_{\lambda\to\infty} z\phi_\mu(\lambda z,\lambda)+\phi_\lambda(\lambda z,\lambda) = 0$, which then implies $\lim_{\lambda\to\infty}\phi_\lambda(\lambda z,\lambda) = 0$.

Step 2: Since $R(\mu_0,\lambda_0) \geq 1$, there exists a unique z^* such that $\lim_{\lambda\to\infty} R(\lambda z^*,\lambda) = R(\mu_0,\lambda_0)$. Along the level curve λ-z where $R(\lambda z,\lambda) = R(\mu_0,\lambda_0)$, we have $T_1(\mu_0,\lambda_0) = 0$ and $\lim_{z\to z^*} T_1(\lambda z,\lambda) = 0$. By Lemma 7, $T_1(\lambda z,\lambda)$ is first decreasing and increasing with z along the level curve of $R(\lambda z,\lambda)$. Therefore, $T_1(\lambda z,\lambda)$ crosses the x-axis at most twice, once from above and once from below. Since we know that at the limit point z^* of this level curve of $R(\lambda z,\lambda)$, $T_1(\lambda z,\lambda)$ is zero, it must cross zero exactly once and from below. This implies that around point (μ_0,λ_0), $T_1(\lambda z,\lambda)$ is decreasing in z along the level curve of $R(\lambda z,\lambda)$. Thus the Hessian matrix of $S(\mu,\lambda)$ is negative definite by Lemma 7. □

The above lemma implies that S is locally concave at point (B_2, B_2+b_1) when $T_1(B_2, B_2+b_1) = 0$ and $R(B_2, B_2+b_1) \geq 1$. Thus as above, $\widetilde{S}^a(b_1,\alpha)$ is locally concave in (b_1,α).

We have thus proved that if $\overline{S}'(b_1) = 0$, then $\overline{S}''(b_1) < 0$. As mentioned before Proposition 3, this implies that the first-order condition is also sufficient. □

A.6 Proof of Proposition 6

In the relaxed problem, sellers select a queue directly in a hypothetical competitive market. The expected payoff for a seller in this market is the difference between the surplus that he creates and the price of the queue. Suppose that a queue (μ,λ) solves sellers' relaxed problem. If queue (μ,λ) contains buyers of value x_k, then $T_k(\mu,\lambda) = U_k$, where $T_k(\mu,\lambda)$ is given by equations (8) and (9); if queue (μ,λ) does not contain buyers of value x_k, then $T_k(\mu,\lambda) \leq U_k$.

Note that when a seller posts a second-price auction with entry fee, t and attracts queue (μ,λ), then the expected payoff of low-type buyers from visiting this seller is $V_1 = Q_1(\lambda)x_1 - (1-Q_0(\lambda))t = \phi_\mu(\lambda,\lambda)x_1 - (1-Q_0(\lambda))t$, and the expected payoff of a high-type buyer from visiting this seller is $V_2 = V_1 + (x_2 - x_1)\phi_\mu(\mu,\lambda)$, which can be verified directly by considering two different scenarios: a high-type buyer faces no competition from any other buyer types, or he faces no competition from other high-type buyers but does compete with low-type buyers.[37] To summarize, the expected payoffs from a second-price auction with an entry fee are

$$V_1 = \phi_\mu(\lambda,\lambda)x_1 - (1-Q_0(\lambda))t,$$
$$V_2 = (x_2-x_1)\phi_\mu(\mu,\lambda) + \phi_\mu(\lambda,\lambda)x_1 - (1-Q_0(\lambda))t.$$

[37] Alternatively, we can use standard auction theory (see Myerson, 1981) and consider the integral of the trading probability (which in our case is $\phi_\mu(\mu,\lambda)$).

An important observation is that if we set t according to equation (20) in the above equation, then $V_k = T_k(\mu, \lambda)$ for $k = 1, 2$. Thus, buyers' expected payoffs from the auction equal their marginal contribution to surplus, which implies that the solution (μ, λ) to a seller's relaxed problem is also compatible with a second-price auction with entry fee t in the sellers' constrained problem, where compatibility is defined by equation (2).

The reserve price case is similar except for one difference. When $r < x_1$, then things are exactly the same as the case with an entry fee and we have $V_k = T_k(\mu, \lambda)$, where $k = 1, 2$. When $r \in (x_1, x_2)$, which happens only when there are no low-type buyers ($\mu = \lambda$), then $V_1 = 0 \leq U_1$ and $V_2 = T_2(\mu, \lambda) = U_2$, in which case the queue is again compatible with a second-price auction with reserve price r in the sellers' constrained problem. □

A.7 Proof of Proposition 7

The sellers' relaxed problem boils down to a competitive market for buyer types. Therefore, the first welfare theorem applies and the equilibrium is efficient. Since the sellers' constrained problem is equivalent to the sellers' relaxed problem, the directed search equilibrium is also efficient. □

A.8 Proof of Proposition 8

Our proof consists of two parts: i) $r^a < r^b \Leftrightarrow \lambda^a > \lambda^b$, and ii) the queue length λ determines the whole queue uniquely. Denote by V_k^i the expected payoff of x_k buyers from visiting queue i where $k = 1, 2$ and $i = a, b$.

For i), we first prove $r^a < r^b \Rightarrow \lambda^a > \lambda^b$. Suppose otherwise that $\lambda^a \leq \lambda^b$. We distinguish between two cases, $r^a < x_1$ and $r^a \geq x_1$. First, consider the case $r^a \geq x_1$. Then x_1 buyers will not visit the two sellers since their market utility is strictly positive, so that both queues contain only x_2 buyers, and $V_2^a = Q_1(\lambda^a)(x_2 - r^a) > Q_1(\lambda^b)(x_2 - r^a) \geq Q_1(\lambda^b)(x_2 - r^b) = V_2^b$. We then reach a contradiction. Consider next $r^a < x_1$. Then by a similar logic, we have

$$V_1^a = Q_1(\lambda^a)(x_1 - r^a) > \max(Q_1(\lambda^b)(x_1 - r^b), 0) = V_1^b$$

Thus x_1 buyers strictly prefer queue a, which implies that queue b does not contain x_1 buyers. Note that $V_2^a \geq Q_1(\lambda^a)(x_2 - r^a) > Q_1(\lambda^b)(x_2 - r^b) = V_2^b$, where the first inequality is because queue a may contain x_1 buyers and an x_2 buyer may enjoy a positive payoff even when he is not the only buyer showing up, the second inequality follows the same logic as above, and the last equality is because queue b only contains x_2 buyers. Therefore, x_2 buyers also strictly prefer queue a and we reach a contradiction again. The other direction is proved similarly. Thus $r^a < r^b \Leftrightarrow \lambda^a > \lambda^b$.

For ii), suppose otherwise that there are two different queues a and b with the same length λ that are compatible with the auction with reserve price r. Without loss of generality, set $0 \leq \mu^a < \mu^b \leq \lambda$. Note that $V_1^a = V_1^b = Q_1(\lambda)(x - r) \equiv V_1$ and the expected payoff of an x_2 buyer from queue i is $V_2^i = V_1 + \phi_\mu(\mu^i, \lambda)(x_2 - x_1)$ (see the proof of Proposition 6 for the derivation of this equation). If $P_0(\lambda) + P_1(\lambda) < 1$, then $\phi(\mu, \lambda)$ is strictly concave in μ, which implies that $V_2^a > V_2^b$ which gives the desired contradiction. If $P_0(\lambda)+P_1(\lambda) = 1$, then $\phi_\mu(\mu, \lambda) = Q_1(\lambda)$, independent of μ, and $V_2^a = V_2^b$. Note that since $Q_1(\lambda)(x_2 - x_1) = (U_2 - U_1)$, and $r = x_1 - U_1/Q_1(\lambda)$, both λ and r are uniquely determined for given market utilities U_1 and U_2. Thus it is a knife-edge case, and our statement is true for all r expect one special value. But note i) this knife-edge phenomenon only occurs because buyer types are discrete, and ii) even in our discrete buyer type framework, this knife-edge reserve price r will never be adopted by sellers because by either increasing or decreasing the reserve price, sellers can obtain a strictly higher profit. □

A.9 Proof of Proposition 9

The proof is similar to that of Proposition 8 and consists of two parts: i) $t^a < t^b \Leftrightarrow \lambda^a > \lambda^b$, and ii) the queue length λ determines the whole queue uniquely. Note that part ii) is exactly the same as that of Proposition 8 so we only need to consider part i).

For i), we first prove $t^a < t^b \Rightarrow \lambda^a > \lambda^b$. Suppose otherwise that $\lambda^a \leq \lambda^b$. We distinguish two cases, $t^a < 0$ and $t^a \geq 0$. First, consider the case $t^a < 0$ (entry subsidy). As before, denote by V_k^i the expected payoff of x_k buyers from a queue i where $k = 1, 2$ and $i = a, b$. Then, we have

$$\begin{aligned} V_1^a = Q_1(\lambda^a)x_1 - (1 - Q_0(\lambda^a))t^a &> Q_1(\lambda^b)x_1 - (1 - Q_0(\lambda^a))t^a \\ &> Q_1(\lambda^b)x_1 - (1 - Q_0(\lambda^b))t^a \\ &\geq Q_1(\lambda^b)x_1 - (1 - Q_0(\lambda^b))t^b = V_1^b \end{aligned}$$

where the first inequality is because $Q_1(\lambda)$ is strictly decreasing, the second inequality is because $1 - Q_0(\lambda)$ is strictly decreasing and $t^a < 0$, and the final inequality follows from the assumption that $t^a < t^b$. Thus x_1 buyers strictly prefer queue a and queue b does not contain x_2 buyers. However, $V_2^a \geq Q_1(\lambda^a)x_1 - (1 - Q_0(\lambda^a))t^a > Q_1(\lambda^b)x_2 - (1 - Q_0(\lambda^b))t^b = V_2^b$, where the first inequality is because queue a may contain x_1 buyers and an x_2 buyer may enjoy a positive payoff even when he is not the only buyer showing up, the second inequality follows the same logic as above, and the last equality is because queue b only contains x_2 buyers. Thus, x_2 buyers also strictly prefer queue a, and we have a contradiction, which implies that $\lambda^a > \lambda^b$. The other direction is proved similarly.

Next, we consider the case $t^a \geq 0$. Again suppose otherwise that $\lambda^a \leq \lambda^b$. As above, we have

$$V_1^a = Q_1(\lambda^a)x_1 - (1 - Q_0(\lambda^a))t^a = Q_1(\lambda^a)\left(x_1 - \frac{1 - Q_0(\lambda^a)}{Q_1(\lambda^a)}t^a\right) > Q_1(\lambda^b)\left(x_1 - \frac{1 - Q_0(\lambda^a)}{Q_1(\lambda^a)}t^a\right)$$
$$\geq Q_1(\lambda^b)\left(x_1 - \frac{1 - Q_0(\lambda^b)}{Q_1(\lambda^b)}t^a\right) > Q_1(\lambda^b)\left(x_1 - \frac{1 - Q_0(\lambda^b)}{Q_1(\lambda^b)}t^b\right) = V_1^b$$

where the inequality in the first line is because $Q_1(\lambda)$ is strictly decreasing, the first inequality in the second line is because of Assumption 4, and the second inequality in the second line follows from the assumption that $t^a < t^b$. The remaining arguments then follow exactly the same as for the case $t^a < 0$. $\square$

References

Agrawal, A., Horton, J., Lacetera, N., and Lyons, E. (2015). Digitization and the contract labor market: A research agenda. In Goldfarb, A., Greenstein, S. M., and Tucker, C. E., editors, *Economic Analysis of the Digital Economy*, pages 219–250. University of Chicago Press.

Albrecht, J. W., Gautier, P. A., and Vroman, S. B. (2012). A note on Peters and Severinov, "Competition among sellers who offer auctions instead of prices". *Journal of Economic Theory*, 147:389–392.

Albrecht, J. W., Gautier, P. A., and Vroman, S. B. (2014). Efficient entry with competing auctions. *American Economic Review*, 104(10):3288–3296.

Albrecht, J. W. and Jovanovic, B. (1986). The efficiency of search under competition and monopsony. *Journal of Political Economy*, 94(6):1246–1257.

Auster, S. and Gottardi, P. (2017). Competing mechanisms in markets for lemons. mimeo.

Backus, M., Blake, T., Masterov, D. V., and Tadelis, S. (2015). Is sniping a problem for online auction markets? NBER Working Paper 20942.

Barro, R. J. and Romer, P. M. (1987). Ski-lift pricing, with applications to labor and other markets. *The American Economic Review*, 77(5):875–890.

Burdett, K., Shi, S., and Wright, R. (2001). Pricing and matching with frictions. *Journal of Political Economy*, 109:1060–1085.

Cai, X., Gautier, P. A., and Wolthoff, R. P. (2017). Search frictions, competing mechanisms and optimal market segmentation. *Journal of Economic Theory*, 169:453 – 473.

Cai, X., Gautier, P. A., and Wolthoff, R. P. (2018). Search, screening and sorting. mimeo.

Davis, S. J. and de la Parra, B. S. (2017). Application flows. mimeo.

Eeckhout, J. and Kircher, P. (2010a). Sorting and decentralized price competition. *Econometrica*, 78:539–574.

Eeckhout, J. and Kircher, P. (2010b). Sorting vs screening - search frictions and competing mechanisms. *Journal of Economic Theory*, 145:1354–1385.

Eggleston, H. G. (1958). *Convexity.* Cambridge University Press.

Einav, L., Farronato, C., Levin, J. D., and Sundaresan, N. (2017). Sales mechanisms in online markets: What happened to internet auctions? *Journal of Political Economy*, forthcoming.

Epstein, L. and Peters, M. (1999). A revelation principle for competing mechanisms. *Journal of Economic Theory*, 88(1):119–161.

Geertz, C. (1978). The bazaar economy: Information and search in peasant marketing. *The American Economic Review*, 68(2):28–32.

Guerrieri, V., Shimer, R., and Wright, R. (2010). Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862. mimeo.

Han, L. and Strange, W. C. (2014). Bidding wars for houses. *Real Estate Economics*, 42(1):1–32.

Korkki, P. (2014). Small business, joining a parade of outsourcing. *New York Times.*

Lester, B., Visschers, L., and Wolthoff, R. (2015). Meeting technologies and optimal trading mechanisms in competitive search markets. *Journal of Economic Theory*, 155:1–15.

Levin, D. and Smith, J. (1994). Equilibrium in auctions with entry. *The American Economic Review*, pages 585–599.

Lucking-Reiley, D. (2000). Auctions on the internet: What's being auctioned, amd how? *Journal of Industrial Economics*, 48(3):227–252.

McAfee, R. P. (1993). Mechanism design by competing sellers. *Econometrica*, 61(6):pp. 1281–1312.

Menzio, G. and Shi, S. (2011). Efficient search on the job and the business cycle. *Journal of Political Economy*, 119:468–510.

Moen, E. R. (1997). Competitive search equilibrium. *Journal of Political Economy*, 105:385–411.

Montgomery, J. D. (1991). Equilibrium wage dispersion and interindustry wage differentials. *Quarterly Journal of Economics*, 106:163–179.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73.

Peters, M. (1997). A competitive distribution of auctions. *The Review of Economic Studies*, 64(1):pp. 97–123.

Peters, M. (2001). Common agency and the revelation principle. *Econometrica*, 69(5):1349–1372.

Peters, M. and Severinov, S. (1997). Competition among sellers who offer auctions instead of prices. *Journal of Economic Theory*, 75:141–179.

Rockafellar, R. T. (1970). *Convex Analysis.* Princeton University Press.

Shimer, R. (2005). The assignment of workers to jobs in an economy with coordination frictions. *Journal of Political Economy*, 113(5):996–1025.

Wolthoff, R. P. (2018). Applications and interviews: Firms' recruiting decisions in a frictional labor market. *Review of Economic Studies*, 85(2):1314–1351.

Meetings and Mechanisms

Online Appendix (not for publication)

Xiaoming Cai* Pieter Gautier† Ronald Wolthoff‡

June 9, 2021

B Proofs

B.1 Proof of Lemma 3

For later use, we prove a slightly more general version of equation (22) with a general, possibly continuous, buyer value distribution.

Lemma 11. *Consider a submarket with a measure 1 of sellers and a measure λ of buyers whose values are distributed according to $F(x)$ with support $[0, \overline{x}]$. Total surplus then equals*

$$S(\lambda, F) = \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) dx. \tag{29}$$

Proof. **A direct proof.** When a seller meets $n \geq 1$ buyers, the surplus x from the meeting is distributed according to $F^n(x)$. Thus the expected surplus per seller in the submarket is

$$S = \sum_{n=1}^{\infty} P_n(\lambda) \int_0^{\overline{x}} x \, dF^n(x) = \sum_{n=1}^{\infty} P_n(\lambda) \left(\overline{x} - \int_0^{\overline{x}} F^n(x) \, dx \right) = \sum_{n=1}^{\infty} P_n(\lambda) \left(\int_0^{\overline{x}} 1 - F^n(x) \, dx \right),$$

where for the second equality we used integration by parts. Notice that $F^n(x) = 0$ when $n = 0$. We can add a zero term $P_0(\lambda) (\int_0^{\overline{x}} 1 - F^0(x) \, dx)$ to the RHS of the above equation and start the summation from $n = 0$. Therefore,

$$\begin{aligned} S = \sum_{n=0}^{\infty} P_n(\lambda) \left(\int_0^{\overline{x}} 1 - F^n(x) \, dx \right) &= \int_0^{\overline{x}} 1 - \sum_{n=0}^{\infty} P_n(\lambda) F^n(x) \, dx \\ &= \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) dx \end{aligned}$$

where for the second equality in the first line we use the Dominated Convergence Theorem to interchange integration with summation and for the last equality we used the definition of ϕ from equation (4).

An alternative approach. First recall the following fact from probability theory. Suppose z is any random variable with cdf $G(z)$ and $z \in [0, \overline{x}]$. Then the expected value of z can be written as

*Peking University HSBC Business School. *E-mail:* xmingcai@gmail.com.
†Vrije Universiteit Amsterdam and Tinbergen Institute. *E-mail:* p.a.gautier@vu.nl.
‡University of Toronto. *E-mail:* ronald.p.wolthoff@gmail.com.

$\mathbb{E}z = \int_0^{\overline{x}} zdG(z) = \int_0^{\overline{x}} 1 - G(z)dz$. This equation is well-known and can be proved by integration by parts. We can use it to directly derive our surplus equation.

(Back to our surplus equation.) Let z be the highest valuation among all buyers that a seller meets. The event $z \geq x$ happens if and only if the seller meets at least one buyer with valuation higher than x, the probability of which is simply $\phi(\lambda(1 - F(x)), \lambda)$ by the construction of ϕ. Therefore, by the above equation we have $S = \mathbb{E}z = \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda)dx$ $\square$

In our discrete case, $F(x_j) = \mu_j/\mu_1$ for $j = 1, \ldots, N$. The above equation reduces to (22).

Next, we calculate $T_k(\boldsymbol{\mu})$. Note that a marginal entrant of x_k buyers increases μ_j, $j = 1, \ldots, k$, by the same amount. Therefore,

$$T_k(\boldsymbol{\mu}) = \sum_{j=1}^{k} \frac{\partial S(\boldsymbol{\mu})}{\partial \mu_j} = \sum_{j=1}^{k} (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1) + \sum_{j=1}^{N} (x_j - x_{j-1})\phi_\lambda(\mu_j, \mu_1)$$

Because total surplus function is constant-returns-to-scale, if we add one more seller and λ more buyers to the submarket while keeping the buyer value distribution unchanged, the added surplus is simply $S(\lambda, F)$ in equation (29). Thus the effect of adding one more seller only is

$$\begin{aligned} R = S - \lambda \frac{\partial S(\lambda, F)}{\partial \lambda} &= \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) - \lambda \frac{\partial \phi(\lambda(1 - F(x)), \lambda)}{\partial \lambda} dx \\ &= \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) - \lambda(1 - F(x))\phi_\mu(\lambda(1 - F(x)), \lambda) - \lambda\phi_\lambda(\lambda(1 - F(x)), \lambda)dx \end{aligned}$$

which is simply equation (24) in the discrete-value case.

Finally, we consider the Hessian matrix. We denote it by $\mathcal{H}$ and its negative is then $-\mathcal{H}$. Also to save space, define $\phi^k_{\mu\mu} \equiv \phi_{\mu\mu}(\mu_k, \mu_1)$, $\phi^k_{\mu\lambda} \equiv \phi_{\mu\lambda}(\mu_k, \mu_1)$, and $\phi^k_{\lambda\lambda} \equiv \phi_{\lambda\lambda}(\mu_k, \mu_1)$ for $k = 1, \ldots, N$. We compute the Hessian matrix by directly calculating $\pi_{ij} \equiv \partial\pi(\boldsymbol{\mu})/\partial\mu_i\mu_j$. Thus we have

$$-\mathcal{H} = \begin{pmatrix} P_0''(\mu_1)x_1 - \sum_2^N \phi_{\lambda\lambda}(\mu_k, \mu_1)(x_k - x_{k-1}), & -\phi^2_{\mu\lambda}(x_2 - x_1), & \cdots & , -\phi^N_{\mu\lambda}(x_N - x_{N-1}) \\ -\phi^2_{\mu\lambda}(x_2 - x_1), & -\phi^2_{\mu\mu}(x_2 - x_1), & \cdots & , 0 \\ \vdots & \vdots, & \cdots & , \vdots \\ -\phi^N_{\mu\lambda}(x_N - x_{N-1}), & 0, & \cdots, & -\phi^N_{\mu\mu}(x_N - x_{N-1}) \end{pmatrix}$$

By Sylvester's criterion, $-\mathcal{H}$ is positive semidefinite if and only if the determinants of the following N matrices are positive: the bottom right 1×1 corner, the bottom right 2×2 corner, ..., and $-\mathcal{H}$ itself. It is easy to see that the bottom right $n \times n$ corner with $n < N$ is always diagonal and the diagonal elements are always positive since $\phi(\mu, \lambda)$ is always concave in μ. Therefore, $-\mathcal{H}$ is positive semidefinite if and only if its determinant is positive.

To calculate the determinant of $-\mathcal{H}$, for each $n \geq 2$ we multiply column n by $-\phi^n_{\mu\lambda}/\phi^n_{\mu\mu}$ and add it to column 1. The resulting matrix is

$$\begin{pmatrix} -\pi_{11} + \sum_2^N \frac{(\phi^k_{\mu\lambda})^2}{\phi^k_{\mu\mu}}(x_k - x_{k-1}), & -\phi^2_{\mu\lambda}(x_2 - x_1), & \cdots & , -\phi^N_{\mu\lambda}(x_N - x_{N-1}) \\ 0, & -\phi^2_{\mu\mu}(x_2 - x_1), & \cdots & , 0 \\ \vdots & \vdots, & \cdots & , \vdots \\ 0, & 0, & \cdots, & -\phi^N_{\mu\mu}(x_N - x_{N-1}) \end{pmatrix}$$

In this way, the matrix $-\mathcal{H}$ becomes upper triangular and its determinant can be easily calculated.

The determinant is

$$\det(-\mathcal{H}) = \left(P_0''(\mu_1)x_1 + \sum_2^N \left(\frac{(\phi_{\mu\lambda}^k)^2}{\phi_{\mu\mu}^k} - \phi_{\lambda\lambda}^k\right)(x_k - x_{k-1})\right)\Pi_2^N(-\phi_{\mu\mu}^k(x_k - x_{k-1}))).$$

Again since $\phi(\mu,\lambda)$ is always concave in μ, $\det(-\mathcal{H}) > 0$ is equivalent to that the first term in the parenthesis at the right hand side is positive. Thus we have derived equation (25).□

B.2 Proof of Proposition 10

Again suppose that at the optimum, there are L active submarkets, and the lowest buyer type is $\underline{x}^\ell$ and the highest buyer type is $\overline{x}^\ell$. Suppose that the marginal contribution to surplus of x_k buyers is T_k^* for $k = 1, 2, \ldots, N$. Thus $T_k^* = \max(\max_{\ell=1,\ldots,L} T_k(\boldsymbol{\mu}^\ell), 0)$, where $T_k(\boldsymbol{\mu}^\ell)$ is the marginal contribution to surplus of x_k buyers in submarket ℓ and is given by equation (23). In the following we write it as T_k^ℓ to simplify notations. Of course, if submarket ℓ contains x_k buyers at the optimum, we must have $T_k^* = T_k^\ell$.

Step 1: Since $\phi(\mu,\lambda)$ is always concave in μ, by equation (23) T_k^ℓ is convex in x_k for each ℓ in the following sense: $\frac{T_2^\ell - T_1^\ell}{x_2 - x_1} \le \frac{T_3^\ell - T_2^\ell}{x_3 - x_2} \le \cdots \le \frac{T_N^\ell - T_{N-1}^\ell}{x_N - x_{N-1}}$ (if buyer types are continuous, then we would have the usual notion of convexity). Since T_k^* is the maximum of a collection of convex functions, it is also convex.

Next, define k_0 as the largest index k such that $T_k^* = 0$, or set $k_0 = 1$ if $T_1^* > 0$. Then we show that T_k^* is strictly convex in the following sense: For $k > k_0$ we have

$$\frac{T_{k+1}^* - T_k^*}{x_{k+1} - x_k} > \frac{T_k^* - T_{k-1}^*}{x_k - x_{k-1}}.$$

To see this, note that since $T_k^* > 0$ for any $i > i_0$, buyers of value x_k must visit some submarket ℓ in which $\mu_k^\ell > \mu_{k+1}^\ell$, i.e., the queue length of buyers with value x_k must be strictly positive in the submarket. In this case, $\phi_\mu(\mu_k^\ell, \mu_1^\ell) < \phi_\mu(\mu_{k+1}^\ell, \mu_1^\ell)$, $T_k^\ell = T_k^*$ and $T_{k+1}^* \ge T_{k+1}^\ell$ and $T_{k-1}^* \ge T_{k-1}^\ell$, which implies that

$$\frac{T_{k+1}^* - T_k^*}{x_{k+1} - x_k} \ge \frac{T_{k+1}^\ell - T_k^\ell}{x_{k+1} - x_k} = \phi_\mu(\mu_{k+1}^\ell, \mu_1^\ell) > \phi_\mu(\mu_k^\ell, \mu_1) = \frac{T_k^\ell - T_{k-1}^\ell}{x_k - x_{k-1}} \ge \frac{T_k^* - T_{k-1}^*}{x_k - x_{k-1}}.$$

Hence we have showed that T_k^* is strictly convex when $i \ge i_0$.

Step 2: Claim: If $\mu_1^a > \mu_1^b$, then $\underline{x}^a \le \underline{x}^b$ and $\overline{x}^a \le \overline{x}^b$. To see this, recall that $Q_1(\mu_1) = \phi_\mu(\mu_1, \mu_1)$ by equation (6). By Assumption 2, $\mu_1^a > \mu_1^b$ implies $Q_1(\mu_1^b) > Q_1(\mu_1^a)$. Note that by equation (23), that $Q_1(\mu_1^\ell)$ is the slope of a supporting line (subgradient) for the function T_k^ℓ and hence the function T_k^* at point $\underline{x}^\ell$ for $\ell \in \{a, b\}$. Because of the strict convexity of T_k^* (see Step 1 of the proof), the subgradient determines point $\underline{x}^\ell$ uniquely, and $Q_1(\mu_1^b) > Q_1(\mu_1^a)$ implies $\underline{x}^b \ge \underline{x}^a$.

Similarly, recall that $1 - Q_0(\mu_1) = \phi_\mu(0, \mu_1)$ by equation (6). By Assumption 2, $\mu_1^a > \mu_1^b$ implies $1 - Q_0(\mu_1^a) > 1 - Q_0(\mu_1^b)$. Note that $1 - Q_0(\mu_1^\ell)$ is a subgradient for the function T_k^ℓ and hence the function T_k^* at point $\overline{x}^\ell$ for $\ell \in \{a, b\}$. As before, the subgradient determines point $\overline{x}^\ell$ uniquely, and $1 - Q_0(\mu_1^a) < 1 - Q_0(\mu_1^b)$ implies $\overline{x}^a \le \overline{x}^b$.

Step 3: Claim: Suppose a submarket ℓ contains buyers of x_{k_1} and x_{k_2} with $k_2 > k_1 + 1$, then it must also contain buyers in between, i.e., buyers of value x_k with $k_1 < k < k_2$. To see this, suppose otherwise (without loss of generality) that submarket ℓ contains no buyers with values between x_{k_1} and x_{k_2}. Then $\mu_{k_1+1} = \cdots = \mu_{k_2}$, which implies that T_k^ℓ is a linear function between x_{k_1} and x_{k_2}. We also know i) $T_k^\ell = T_k^*$ for $k = k_1$ and k_2, and ii) from Step 1 that T_k^* is a strictly convex function. The above two observations imply that $T_k^\ell > T_k^*$ for $k_1 < k < k_2$, which then leads to a contradiction.

Step 4: If $\overline{x}^a \leq \underline{x}^b$, then the proposition is true automatically. In the following, we will thus assume $\underline{x}^b < \overline{x}^a$. Therefore, we have $\underline{x}^a \leq \underline{x}^b < \overline{x}^a \leq \overline{x}^b$. We consider some x_k with $\underline{x}^b < x_k \leq \overline{x}^a$. By equation (23), $T_k^\ell = T_{k-1}^\ell + \phi_\mu(\mu_k^\ell, \mu_1^\ell)(x_k - x_{k-1})$ for $\ell \in \{a, b\}$. Note that x_k and x_{k-1} buyers visit both submarket a and b by Step 3, then $T_k^\ell = T_k^*$ and $T_{k-1}^\ell = T_{k-1}^*$ for $\ell = a$ or b. Therefore, we have

$$\phi_\mu\left(\mu_k^a, \mu_1^a\right) = \phi_\mu\left(\mu_k^b, \mu_1^b\right). \tag{30}$$

We then prove the claim by contradiction. Suppose that $\mu_k^b/\mu_1^b < \mu_k^a/\mu_1^a$ for $\underline{x}^b < x_k \leq \overline{x}^a$. This implies

$$\phi_\mu\left(\mu_k^a, \mu_1^a\right) < \phi_\mu\left(\mu_1^a \frac{\mu_k^b}{\mu_1^b}, \mu_1^a\right) < \phi_\mu\left(\mu_1^b \frac{\mu_k^b}{\mu_1^b}, \mu_1^b\right) = \phi_\mu\left(\mu_k^b, \mu_1^b\right),$$

where the first inequality is because $\phi\left(\mu, \mu_1^a\right)$ is strictly concave in μ and the second is because of assumption 2. The above inequality is at odds with equation (30). Hence, we have reached a contradiction. □

B.3 Boundary.

Submarket with Only Low-Type Buyers. Consider a submarket that has a queue λ with only low-type buyers. Then, the marginal contribution of these buyers is $T_1(0,\lambda) = m'(\lambda)$, while sellers' marginal contribution is $R(0,\lambda) = m(\lambda) - \lambda m'(\lambda)$. For future reference, we can therefore define a function g which maps the marginal contribution to surplus of sellers to that of low-type buyers. That is, for any $\lambda > 0$, we have

$$T_1(0,\lambda) = g(R(0,\lambda)). \tag{31}$$

Alternatively, we can define g explicitly as

$$g(R) = \begin{cases} m'\left((m - \lambda m')^{-1}(R)\right) & \text{for } R \in [0,1) \\ 0 & \text{for } R \geq 1, \end{cases}$$

where $(m - \lambda m')^{-1}$ is the inverse function of $m - \lambda m'$. Since $\frac{d}{d\lambda} R(0,\lambda) = -\lambda m''(\lambda)$ and $\frac{d}{d\lambda} T_1(0,\lambda) = -m''(\lambda)$, we have

$$g'(R) = -\frac{1}{\lambda} \quad \text{if} \quad R = m(\lambda) - \lambda m'(\lambda). \tag{32}$$

When $R \geq 1$, we have $g'(R) = 0$. For the geometrically-truncated-geometric meeting technology with $m(\lambda) = \lambda/(1+\lambda)$, one can verify that $g(R) = (1-\sqrt{R})^2$ when $R \in [0,1)$ and $g(R) = 0$ for $R \geq 1$.

Proposition 3 tells us that pooling is optimal if $\overline{S}'(B_1) \geq 0$, i.e. the marginal contribution of low-type buyers is greater in the segment with the high-type buyers than in a separate segment with an ε amount of low-type buyers and where sellers are optimally allocated. In the following, we are especially interested in the cutting-edge case where $\overline{S}'(B_1)$ in equation (17) is exactly zero. That is, $(B_2, B_1 + B_2)$ is a solution to the following equation

$$T_1(\mu,\lambda) = g(R(\mu,\lambda)). \tag{33}$$

In this case, the planner's solution is pooling, but $(B_2, B_1 + B_2)$ lies on the boundary of the pooling area.

Among the solutions to equation (33), we distinguish between two cases: i) $R(\mu,\lambda) < 1$, and ii) $R(\mu,\lambda) \geq 1$, which implies $T_1(\mu,\lambda) = g(R(\mu,\lambda)) = 0$. In the first case, by the definition of g, there exists a λ_0 such that $R(\mu,\lambda) = R(0,\lambda_0)$ and $T_1(\mu,\lambda) = T_1(0,\lambda_0)$, which, by equations (8) and (10), implies that

$$\frac{m(\lambda_0) - m(\lambda) - (\lambda_0 - \lambda)m'(\lambda_0)}{m'(\lambda) - m'(\lambda_0)} = \frac{\phi(\mu,\lambda) - \mu\phi_\mu(\mu,\lambda)}{-\phi_\lambda(\mu,\lambda)} \tag{34}$$

By taking the derivative with respect to λ_0, one can see that the right-hand side above is strictly increasing in λ with its infimum being 0 as $\lambda_0 \to \lambda$ and its supremum being $(1 - m(\lambda))/m'(\lambda)$, the Mills ratio of function $m(\lambda)$, as $\lambda_0 \to \infty$. For future use, we introduce a new function to denote the solution of λ_0 to the above equation. Specifically, we define

$$\Lambda(\mu,\lambda) = \begin{cases} \lambda_0 & \text{the solution in equation (34),} \quad \text{if} \quad \frac{\phi(\mu,\lambda)-\mu\phi_\mu(\mu,\lambda)}{-\phi_\lambda(\mu,\lambda)} \in \left(0, \frac{1-m(\lambda)}{m'(\lambda)}\right) \\ \infty, & \text{if} \quad \frac{\phi(\mu,\lambda)-\mu\phi_\mu(\mu,\lambda)}{-\phi_\lambda(\mu,\lambda)} \geq \frac{1-m(\lambda)}{m'(\lambda)} \end{cases} \tag{35}$$

Note that $\Lambda(\mu,\lambda)$ is not well-defined at points (μ,λ) where $\phi_\lambda(\mu,\lambda) > 0$, i.e. the meeting externalities are positive. To simplify exposition and to focus on the more realistic case where buyers crowd each other out, we will therefore sometimes impose the following assumption.

Assumption 5. $\phi_\lambda(\mu,\lambda) < 0$ *for* $0 < \mu \leq \lambda$.

Note that the inequality in this assumption is strict, which means that it is satisfied by the geometrically truncated geometric technology as long as $\sigma < 1$. Cases in which $\phi_\lambda(\mu,\lambda) = 0$ can be analyzed separately, as then Assumptions 1, 2, and 3 are satisfied automatically. Therefore, $\overline{S}'(B_1) = T_1(B_2, B_2 + B_1) - g(R(B_2, B_2 + B_1)) \geq m'(B_2 + B_1) - g(R(0, B_2 + B_1)) = 0$, where the inequality is because (i) $R(B_2, B_2 + B_1) \geq R(0, B_2 + B_1)$ and (ii) g is decreasing. Therefore, Proposition 3 implies that the planner's solution is always pooling because $\overline{S}'(B_1) \geq 0$. Since this special case is easy and to avoid the issue of division by zero, we adopt Assumption 5 for the following analysis of comparative statics. Finally, note that for any meeting technology, $\phi(0,\lambda) = 0$, which implies that $\phi_\lambda(0,\lambda) = 0$ for any λ. Hence in the above assumption we require $\mu > 0$. □

B.4 Special Cases

B.4.1 Geometric Technology Truncated at 2

Consider the deterministically truncated geometric meeting technology with a capacity of 2. In this case, the function ϕ is given by

$$\phi(\mu,\lambda) = \frac{\mu(1 + 2\lambda - \mu)}{(1+\lambda)^2} \tag{36}$$

This is arguably the simplest meeting technology that is neither bilateral nor invariant. By equations (10) and (8), we have

$$R(\mu,\lambda) = \frac{2(x_2 - 1)\lambda^2\mu - (x_2 - 1)(\lambda - 1)\mu^2 + (\lambda + 1)\lambda^2}{(\lambda + 1)^3} \tag{37}$$

$$T_1(\mu,\lambda) = \frac{1}{(1+\lambda)^2} - (x_2 - 1)\frac{2\mu(\lambda - \mu)}{(1+\lambda)^3} \tag{38}$$

To solve the planner's problem, we first calculate the function $H(\mu,\lambda)$ defined by equation (12),

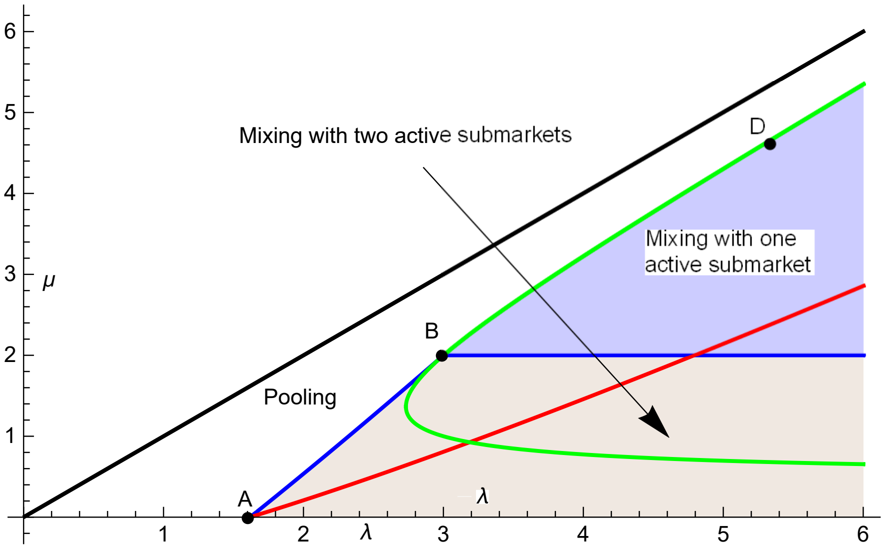


Figure 1: GC2 with $(x_2 - 1) = 1$

which is given by

$$H(\mu, \lambda) = \frac{(\lambda - \mu)^2 - \mu}{1 + \lambda}$$

This is a quadratic equation in μ. Note that $H(0, \lambda) > 0$ and $H(\lambda, \lambda) < 0$, which implies that the equation $H(\mu, \lambda) = 1/(x_2 - 1)$ has a unique root of μ, which is given by,

$$\mu = h(\lambda) \equiv \frac{1 + 2\lambda}{2} - \frac{1}{2}\sqrt{1 + 4\lambda + 4\frac{1 + \lambda}{(x_2 - 1)}} \quad \text{with} \quad \lambda \geq \lambda_A \tag{39}$$

where λ_A is the root of $h(\lambda) = 0$ and is given by

$$\lambda_A = \frac{1 + \sqrt{1 + 4(x_2 - 1)}}{2(x_2 - 1)}. \tag{40}$$

The curve $h(\lambda)$ is represented by the red curve in Figure 1 where we set $(x_2 - 1) = 1$. The Hessian matrix is negative definitive in the area to left of the red curve; the surplus function $S(\mu, \lambda)$ is locally concave at any point in this area.

Next, we proceed to solve equation (33). First, consider the equation $T_1(\mu, \lambda) = 0$. By equation (38), for any given μ there exists a unique λ such that $T_1(\mu, \lambda) = 0$ (the converse is false), and the solution is simply

$$\lambda = \mu + \frac{1 + \mu}{2(x_2 - 1)\mu - 1}, \qquad \text{where } \mu > \frac{1}{2(x_2 - 1)} \tag{41}$$

The above function is represented by the green curve in Figure 1. Note that the upper branch approaches asymptotically to the diagonal line, and the lower branch approaches asymptotically to the horizontal line $\mu = 1/(2(x_2 - 1))$. When $\mu < 1/(2(x_2 - 1))$, $T_1(\mu, \lambda)$ is always positive. A point (μ_0, λ_0) satisfies $T_1(\mu_0, \lambda_0) \geq 0$ if and only if it lies to the left of the green curve.

Next, note that

$$R(0,\lambda) = m(\lambda) - \lambda m'(\lambda) = \left(\frac{\lambda}{1+\lambda}\right)^2, \quad \text{and} \quad T_1(0,\lambda) = m'(\lambda) = \left(\frac{1}{1+\lambda}\right)^2$$

which implies that

$$g(R) = (1-\sqrt{R})^2 \tag{42}$$

We plug equations (37), (38), and (42) into equation (33). After some rearrangements, we can find the solution, which is given by the following.

$$\lambda = \frac{1+(x_2-1)\mu + \sqrt{1+4(x_2-1)+2(x_2-1)\mu}}{2(x_2-1)}, \quad \text{where } 0 \le \mu \le \mu_B \tag{43}$$

and

$$\mu_B = \frac{3+\sqrt{9+16(x_2-1)}}{4(x_2-1)}$$

The above function, (43), is represented by the solid blue curve AB in Figure 1. It intersects with the curve $T_1(\mu,\lambda) = 0$, equation (41), at point (λ_B, μ_B), which is represented by Point B in Figure 1. At Point B, $T_1(\mu_B, \lambda_B) = 0$ and $R(\mu_B, \lambda_B) = 1$.

Setting $\mu = 0$ in equation (43) gives $\lambda = \frac{1+\sqrt{1+4(x_2-1)}}{2(x_2-1)}$, which is exactly point A from equation (40), the same point where $h(\lambda)$ crosses the x-axis. This is no coincidence; it holds for general meeting technologies. To see why, consider point S_1 that lies on the curve AB in Figure 1. It has a corresponding point S_3 on the x-axis. The marginal contributions to surplus of sellers and of x_1 buyers are the same between the two points. Therefore, by Lemma 7, S_1 must lie to the left of the red curve where the determinant of the Hessian matrix is 0. Since S_1 is an arbitrary point on the curve AB, this implies that the entire curve AB must lie to the left of the red curve. As point S_3 moves towards point A, S_2 and hence S_1 also move towards point A. In the end, all three points coincide at point A, which then implies that the blue and the red curve intersect at the same point on the x-axis.

The above analysis has pinned down the boundaries of the relevant regions. The planner's solution is summarized by the following. Note that they satisfy the first-order conditions and are hence optimal by Proposition 3.

Suppose that $(B_1 + B_2, B_2)$ belongs to the blue area. Assume pooling initially, then the marginal contribution of x_1 buyers is negative. Therefore, the planner will move x_1 buyers to a second submarket and the queue in the first submarket will move horizontally to the left till it reaches the green curve BD. At that point, there is one active submarket where the marginal contribution of x_1 buyers is 0 and the marginal contribution of sellers is larger than 1, and one idle submarket with only x_1 buyers.

Suppose $(B_1 + B_2, B_2)$ belongs to the brown area. As we mentioned before, for each point (λ, μ) on the curve AB, there is a corresponding point on the x-axis such that R and T_1 are the same between the two. Formally, the point is given by $(\Lambda(\mu,\lambda), 0)$, where $\Lambda(\mu,\lambda)$ is defined by equation (35). As we move from point A to point B, the corresponding point on the x-axis moves from point A to infinity. The convex combinations between points on AB and their corresponding point on x-axis cover the whole brown area. For each point in the brown area, after representing it as a convex combination between a point on AB and its corresponding point on the x-axis, the first-order condition of the planner's problem is satisfied by construction, and we have the optimum: two active submarkets where the queue in the first submarket must lie on the AB curve (for example point S_1) and the second submarket contains some sellers and x_1 buyers (for example point S_3).

If $(B_1 + B_2, B_2)$ belongs to the white area, then the optimum is pooling. Note that curve AB divides the area where $T_1(\mu, \lambda) \geq 0$ into two disconnected areas: $T_1(\mu, \lambda) > g(R(\mu, \lambda))$ and $T_1(\mu, \lambda) < g(R(\mu, \lambda))$, with curve AB being the boundary. The white area is the former, and it is not socially beneficial to even move an ε amount of x_1 buyers to a second submarket. Thus the optimum is pooling.

B.4.2 Example: Geometrically Truncated Geometric Technology

Before solving the planner's problem, we first calculate $\phi(\mu, \lambda)$, the result of which is given by the following lemma.

Lemma 12. *For the geometrically truncated geometric technology, we have*

$$\phi(\mu, \lambda) = \frac{\mu}{1 + \sigma\mu + (1 - \sigma)\lambda}. \tag{44}$$

Proof. The seller's meeting capacity n_C follows a geometric distribution with support $\mathbb{N}_1$ and mean $(1 - \sigma)^{-1}$. That is, $\mathbb{P}(n_C = n) = (1 - \sigma)\sigma^{n-1}$ for $n = 1, 2, \ldots$. Meanwhile, the number of buyers who visit the seller, n_A, also follows a geometric distribution but with support $\mathbb{N}_0$ and mean λ, i.e., $\mathbb{P}(n_A = n) = \frac{1}{1+\lambda}(\frac{\lambda}{1+\lambda})^n$ for $n = 0, 1, 2, \ldots$. The actual number of buyers that the seller meets, n, is then $\min\{n_C, n_A\} \in \mathbb{N}_0$. Hence $P_n(\lambda) \equiv \mathbb{P}[\min\{n_C, n_A\} = n \,|\, \lambda]$. Since the capacity constraint n_C is at least one, $P_0(\lambda) = \frac{1}{1+\lambda}$. For $n \geq 1$, we have

$$P_n(\lambda) = (1 - \sigma)\sigma^{n-1} \sum_{j=n}^{\infty} \frac{1}{1 + \lambda}\left(\frac{\lambda}{1 + \lambda}\right)^j + \frac{1}{1 + \lambda}\left(\frac{\lambda}{1 + \lambda}\right)^n \sum_{j=n+1}^{\infty} (1 - \sigma)\sigma^{j-1}.$$

The first term on the right-hand side denotes the case where the number of buyers is (weakly) larger than n while the meeting capacity equals n. The second term denotes the case where the number of buyers equals n while the meeting capacity is strictly larger than n. Simplifying the summations yields

$$P_n(\lambda) = \begin{cases} \frac{1}{1+\lambda} & \text{for } n = 0, \\ \sigma^{n-1} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda}\right)^n (1 + (1 - \sigma)\lambda) & \text{for } n \in \mathbb{N}_1. \end{cases} \tag{45}$$

Substituting (45) into equation (4) and simplifying the result yields equation (44). □

We now solve the planner's problem analytically. We show that the outcome depends on the extent to which sellers can meet multiple buyers, as determined by the value of σ. We distinguish between three regions by specifying two cutoff values for σ, i.e. $\sigma_0(x_2)$ and $\sigma_1(x_2)$, defined as

$$\sigma_0(x_2) \equiv \frac{\sqrt{x_2} - 1}{\sqrt{x_2} + 1} < \frac{\sqrt{x_2}}{\sqrt{x_2} + 1} \equiv \sigma_1(x_2) \tag{46}$$

Low Sigma. We first consider the case in which $\sigma \leq \sigma_0(x_2)$. Using the functional form for $\phi(\mu, \lambda)$ given in (44), a straightforward calculation yields

$$H(\mu, \lambda) = \frac{(1 - \sigma)^2}{4\sigma} \frac{(1 + \lambda)^3}{(1 + (1 - \sigma)\lambda)(1 + \sigma\mu + (1 - \sigma)\lambda)} > \frac{(1 - \sigma)^2}{4\sigma} \geq \frac{1}{x_2 - 1}, \tag{47}$$

where the first inequality follows because the second factor in $H(\mu, \lambda)$ is strictly larger than 1, and the second inequality is implied by $\sigma \leq \sigma_0(x_2)$. Consequently, the second-order condition (12) can never be satisfied in this case, i.e. a submarket (μ, λ) where $0 < \mu < \lambda$ cannot be part of the planner's solution. Instead, perfect separation is obtained: one submarket contains all high-type buyers and another submarket contains all low-type buyers.

The allocation of sellers depends on their marginal contribution to surplus, which equals

$$R(\mu,\lambda) = \frac{(x_2-1)\mu(\sigma\mu+(1-\sigma)\lambda)}{(1+\sigma\mu+(1-\sigma)\lambda)^2} + \frac{\lambda^2}{(1+\lambda)^2}. \tag{48}$$

If $R(B_2, B_2) \geq 1$, then the planner will allocate all sellers to the submarket with high-type buyers; otherwise both submarkets will be active.[1]

Intermediate Sigma. We now consider the case $\sigma \in (\sigma_0(x_2), \sigma_1(x_2)]$, which is illustrated in Figure 2a. The key object for determining the planner's solution is the marginal contribution to surplus of low-type buyers, i.e.

$$T_1(\mu,\lambda) = \frac{1}{(1+\lambda)^2} - \frac{(x_2-1)(1-\sigma)\mu}{(1+\sigma\mu+(1-\sigma)\lambda)^2}. \tag{49}$$

First, we are interested in combinations of μ and λ for which $T_1(\mu,\lambda) = 0$, as this is the minimum requirement for a submarket with high-type buyers to also contains low-type buyers. Straightforward algebra shows that, for any $(x_2-1)^{-1}(1-\sigma) \leq \mu \leq (x_2-1)^{-1}(1-\sigma)^{-1}$, there exists a unique λ such that $T_1(\mu,\lambda) = 0$. The locus of these points is represented by the green curves in Figure 2a and 2b. Low-type buyers' contribution to surplus is negative in points above this curve, which therefore cannot be part of the planner's solution.

Next, we are interested in the combinations of μ and λ for which $T_1(\mu,\lambda) = g(R(\mu,\lambda))$, as this is where the planner is indifferent between keeping low-type buyers in the submarket and sending them (with an optimal number of sellers) to a separate submarket. Using (31), (47) and (49) we can solve for μ as a function of λ, i.e.

$$\mu = \frac{\sqrt{(x_2-1)(1+\lambda)}}{\sqrt{(x_2-1)(1+\lambda)} - 2\sqrt{\sigma}} - \frac{1+(1-\sigma)\lambda}{\sigma}. \tag{50}$$

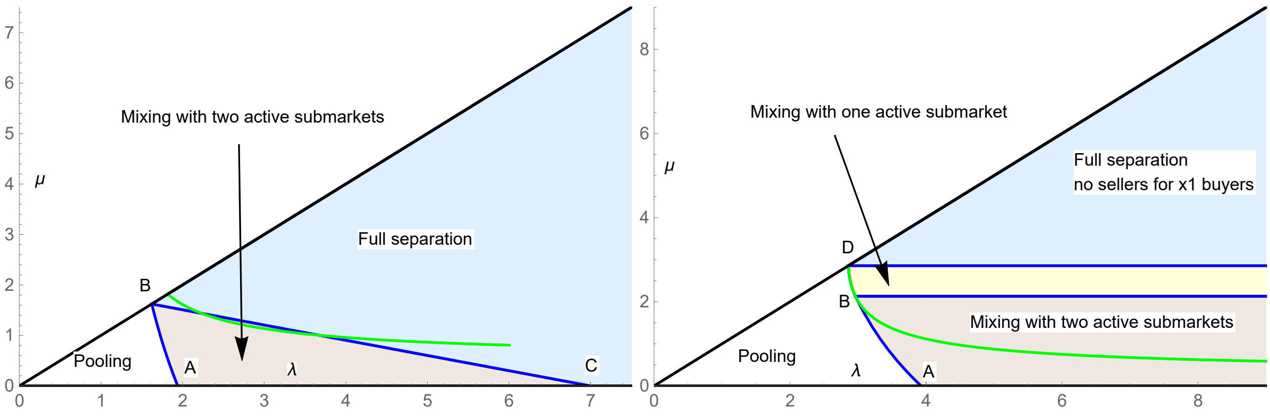
This solution is represented by the solid blue curve AB in Figure 2a; sending some low-type buyers and sellers to form a new submarket is beneficial right but not left of this curve. The end points of the curve, i.e. point $A = (\lambda_A, 0)$ on the horizontal axis and point $B = (\lambda_B, \lambda_B)$ on the diagonal, can be obtained by solving (50) for $\mu = 0$ and $\mu = \lambda$, respectively. The latter yields $\lambda_B = \sigma\frac{(1+\sqrt{x_2})}{\sqrt{x_2}-1} - 1$.

For every point on the segment AB, there exists—by the definition of g—a corresponding point $(\lambda_0, 0)$ on the horizontal axis with the same marginal contributions of sellers and low-type buyers, i.e. $R(\mu,\lambda) = R(0,\lambda_0)$ and $T_1(\mu,\lambda) = T_1(0,\lambda_0)$ where λ_0 is given explicitly by the function $\Lambda(\mu,\lambda)$ defined in (35) in Appendix B.3. As we move from A to B, this corresponding point moves from A to $C = (\lambda_C, 0)$, where $\lambda_C = \Lambda(\lambda_B, \lambda_B)$. The thresholds $\sigma_0(x_2)$ and $\sigma_1(x_2)$ are determined by the location of point C: as $\sigma \searrow \sigma_0(x_2)$, point C approaches $(0, 0)$; in contrast, when $\sigma \nearrow \sigma_1(x_2)$, then $\lambda_C \nearrow \infty$.

We can now characterize the planner's solution. If $(B_2, B_1 + B_2)$ belongs to the brown area in Figure 2a, then the optimum is two active submarkets, where the queue in the first submarket lies on the curve AB, and the queue in the second submarket is the corresponding point $(\Lambda(\mu,\lambda), 0)$ on the horizontal axis between point A and C. If $(B_2, B_1 + B_2)$ belongs to the blue area, then the optimum is full separation where one submarket contains all x_2 buyers and the other contains all x_1 buyers. Whether the submarket with x_1 buyers contains sellers depends on B_2: if $R(B_2, B_2) \geq 1$, then the first

[1] We thus prove analytically that for any given x_2, there exists a meeting technology such that full separation is always optimal for any endowment of buyers. This proves the conjecture in section 5.3 of Eeckhout and Kircher (2010b) who only showed the existence of such a meeting technology numerically.

submarket contains all sellers, otherwise both submarkets contain sellers. Finally, when $(B_2, B_1 + B_2)$ belongs to the white area, then the optimum is pooling where one market contains all sellers and buyers.



(a) $\sigma = 0.45 < \sigma_1(x_2)$ (b) $\sigma = 0.65 > \sigma_1(x_2)$

Figure 2: The planner's solution with $x_2 = 2$

High Sigma. Finally, we consider the case $\sigma > \sigma_1(x_2)$, which is illustrated in Figure 2b. The analysis is quite similar to the case with intermediate sigma. One key difference, however, is that point B no longer lies on the diagonal but on the green curve where $T_1(\mu, \lambda) = 0$. As a result, another point plays an important role: the intersection between the green curve $T_1(\mu, \lambda) = 0$ and the diagonal. This intersection is point D in Figure 2b. Note that at any point on the segment BD of the green curve, we have $T_1(\mu, \lambda) = 0$ and $R(\mu, \lambda) \geq 1$.

After establishing the relevant boundaries, we can then summarize the planner's solution. If $(B_2, B_1 + B_2)$ belongs to the brown area in Figure 2b, then it is optimal to open two active submarkets, where the queue in the first submarket lies on the curve AB, and the second submarket is the corresponding point on the horizontal axis characterized by the same marginal contribution to surplus of sellers and low-type buyers. If $(B_2, B_1 + B_2)$ belongs to the yellow area, then it is optimal to have one active submarket and one inactive submarket. The queue in the first submarket lies on the curve BD and the second contains only low-type buyers and no sellers. If $(B_2, B_1 + B_2)$ belongs to the blue area, then the optimum is full separation where one submarket contains all sellers and x_2 buyers and the other contains all x_1 buyers and no sellers. When $(B_2, B_1 + B_2)$ belongs to the white area, then the optimum is pooling where one market contains all sellers and buyers.

B.5 Comparative Statics

B.5.1 Changes in the Dispersion of Buyer Values

To highlight the dependence of $\overline{S}'(B_1)$, R and T_1 on x_2, we append it to the arguments of these functions and write $\overline{S}'(B_1, x_2)$, etc.

Consider complete pooling first. As established in equation (18), complete pooling is socially optimal if and only if $\overline{S}'(B_1, x_2) \geq 0$. Pooling continues to be optimal for x_2' with $x_2' < x_2$ if and only if $\overline{S}'(B_1, x_2') \geq 0$. Thus, for the pooling area to shrink as x_2 increases, we need that $\overline{S}'(B_1, x_2)$, as a function of x_2, crosses the x-axis at most once and from above. As is well-known in the literature, a sufficient condition for this is that $\frac{\partial}{\partial x_2}\overline{S}'(B_1, x_2) < 0$ if $\overline{S}'(B_1, x_2) = 0$; a necessary condition is that $\frac{\partial}{\partial x_2}\overline{S}'(B_1, x_2) \leq 0$ if $\overline{S}'(B_1, x_2) = 0$ (note the difference between the strict and weak inequality). By equation (17),

$$\frac{\partial}{\partial x_2}\overline{S}'(B_1,x_2) = \frac{\partial}{\partial x_2}T_1(B_2,B_2+B_1,x_2) - g'(R(B_2,B_2+B_1,x_2))\frac{\partial}{\partial x_2}R(B_2,B_2+B_1,x_2)$$
$$= \phi_\lambda + \frac{1}{\Lambda(B_2,B_1+B_2)}\left(\phi - B_2\phi_\mu - (B_1+B_2)\phi_\lambda\right), \tag{51}$$

where we have suppressed argument $(B_2, B_1 + B_2)$ from function ϕ, and function Λ is defined by equation (35). It turns out that equation (35) implies that the above equation is always strictly negative, and we have the following result.

Proposition 9. *Under Assumption 1, 2, 3, and 5, the area in which complete pooling is optimal is shrinking in x_2.*

Note that Assumption 5 holds for the geometrically truncated geometric technology, so the result applies.

Proof. If $R(B_2, B_2 + B_1, x_2) \geq 1$, then $g(R(B_2, B_2 + B_1, x_2)) = 0$. $\overline{S}'(B_1, x_2) = 0$ then implies $T_1(B_2, B_2+B_1, x_2) = g(R(B_2, B_2+B_1, x_2)) = 0$ and $\Lambda(B_2, B_1+B_2) = \infty$. Since $T_1(B_2, B_2+B_1, x_2) = m'(B_2 + B_1) + (x_2 - 1)\phi_\lambda(B_2, B_1 + B_2)$, we have $\phi_\lambda(B_2, B_1 + B_2) < 0$. Thus (51) is strictly negative.

Next, if $R(B_2, B_2 + B_1, x_2) \in (0, 1)$, then $g(R(B_2, B_2 + B_1, x_2)) \in (0, 1)$. Thus $\overline{S}'(B_1, x_2) = 0$ implies that $\Lambda(B_2, B_1 + B_2)$ is defined by first row of equation (35) and we have $\phi_\lambda(B_2, B_1 + B_2) < 0$. It is easy to see that (51) is strictly negative if and only if the following holds

$$\frac{-\phi_\lambda}{\Lambda}\left(\Lambda - (B_1+B_2) - \frac{\phi - B_2\phi_\mu}{-\phi_\lambda}\right)$$
$$= \frac{-\phi_\lambda}{\Lambda}\left(\Lambda - (B_1+B_2) - \frac{m(\Lambda) - m(B_1+B_2) - (\Lambda - (B_1+B_2))m'(\Lambda)}{m'(B_1+B_2) - m'(\Lambda)}\right)$$
$$= \frac{-\phi_\lambda}{\Lambda}\left(\frac{(\Lambda - (B_1+B_2))\,m'(B_1+B_2) - (m(\Lambda) - m(B_1+B_2))}{m'(B_1+B_2) - m'(\Lambda)}\right) < 0.$$

where we have suppressed the arguments of Λ and ϕ. For the equality in the second line we used equation (34), and the last inequality is because m is strictly concave and $\Lambda > B_1 + B_2$. □

Next consider full separation. The logic is similar to the case for pooling. By equation (18), full separation is socially optimal if and only if $\overline{S}'(0, x_2) \leq 0$. For the area of full separation to expand with x_2, we need that $\overline{S}'(0, x_2)$, as a function of x_2, crosses the x-axis at most once and from above.

Assume $\overline{S}'(0, x_2) = 0$. Let λ_H (resp. λ_L) be the queue length in the submarket of high-type (resp. low-type) buyers at the optimum. These queue lengths are determined by two equations. First, sellers' marginal contribution to surplus must be the same between the two submarkets, i.e.

$$x_2\left(m(\lambda_H) - \lambda_H m'(\lambda_H)\right) = m(\lambda_L) - \lambda_L m'(\lambda_L), \tag{52}$$

Second, summing the number of sellers across the two submarkets must yield the total measure of sellers, i.e.

$$\frac{B_2}{\lambda_H} + \frac{B_1}{\lambda_L} = 1. \tag{53}$$

Next, consider the marginal contribution to surplus of low-type buyers in the two submarkets.

Since $\overline{S}'(0, x_2) = 0$, we have

$$m'(\lambda_H) + (x_2 - 1)\phi_\lambda(\lambda_H, \lambda_H) = m'(\lambda_L) \tag{54}$$

where the left- and the right-hand denotes the marginal contribution of a low-type buyer in the submarket with high-type and low-type buyers, respectively.[2]

Now suppose that x_2 increases to x_2'. We want to rule out the possibility that $\overline{S}'(0, x_2') = 0$. Suppose otherwise. Then at x_2', the new queue lengths λ_H' and λ_L' also satisfy equations (52), (53), and (54). Note that equations (53) and (54) are special cases of equation (10) and (8), respectively. As before, we can combine equations (53) and (54) to eliminate x_2 and the resulting equation is simply (34), where the correspondence is $\mu = \lambda = \lambda_H$ and $\lambda_0 = \lambda_L$. Thus we have $\lambda_L = \Lambda(\lambda_H, \lambda_H)$ and $\lambda_L' = \Lambda(\lambda_H', \lambda_H')$, where $\Lambda(\mu, \lambda)$ is defined by equation (35) and is independent of x_2.

Conditional on full separation, the allocation of sellers is completely determined by equation (52). When x_2 increases to x_2', more sellers will vist the submarket with high-type buyers, which then implies that $\lambda_H' < \lambda_H < \lambda_L < \lambda_L'$. To rule out that λ_H' and λ_L' are also a solution to equation (54), a sufficient and necessary condition is simply that $\Lambda(\mu, \mu)$ is weakly increasing in μ, which then implies that if $\lambda_H' < \lambda_H$, then $\lambda_L' = \Lambda(\lambda_H', \lambda_H') \leq \Lambda(\lambda_H, \lambda_H) = \lambda_L$, which contradicts the above assertion that $\lambda_L' > \lambda_L$. This leads to the following assumption.

Assumption 6. $\Lambda(\mu, \mu)$*, which is defined by equation (35), is (weakly) increasing in* μ*.*

In equation (57), we give an explicit expression for $\Lambda(\mu, \lambda)$ for the geometrically truncated geometric meeting technology, which verifies that the above assumption is satisfied.

We then have the following result.

Proposition 10. *Under Assumption 1, 2, 3, 5, and 6, the area in which full separation is optimal is expanding with* x_2*.*

Proof. As we explained before Proposition 9, Assumption 6 ensures that $\overline{S}'(0, x_2)$ crosses the x-axis at most once. We now prove that if that is the case, $\overline{S}'(0, x_2)$ must cross the x-axis from above. Suppose otherwise. Then $\overline{S}'(0, x_2) = 0$ for some x_2, and $\overline{S}'(0, x_2') > 0$ for all $x_2' > x_2$. As we mentioned before, more sellers will flow into the submarket of x_2 buyers as we increase x_2, and there exists some x_2* such that the solution to equations (52) and (53) is $\lambda_H = B_2$ and $\lambda_L = \infty$. If we increase x_2 further, then λ_H will stay constant and the right-hand side of (54) will start decreasing linearly. So $\overline{S}'(0, x_2)$ can not stay positive for sufficiently large x_2, and we have a contradiction. □

B.5.2 Changes in Screening Capacity

Analogous to the geometrically truncated geometric case, we assume that the meeting technology is indexed by a parameter σ. To highlight the dependence of ϕ, R, T_1 and $\overline{S}'(B_1)$ on σ, we append it to the arguments of these functions and write $\phi(\mu, \lambda, \sigma)$, etc. We make the following assumption about how ϕ varies with σ.

Assumption 7. *For any* μ *and* λ*,* $\frac{\partial}{\partial \sigma}\phi(\mu, \lambda, \sigma) \geq 0$*, and* $\frac{\partial}{\partial \sigma}\phi(\lambda, \lambda, \sigma) = 0$*.*

Note that the above assumption holds trivially for the geometrically truncated geometric meeting technology. The first part of this assumption states that a higher σ leads to a higher probability of meeting at least one high-type buyer. The second part states that the probability that a seller meets at least one buyer is independent of σ. In other words, a higher σ makes it easier to identify certain

[2]Note that although the submarket with high-type buyers does not contain low-type buyers, we can still calculate the effect on surplus of a marginal increase in the number of low-type buyers.

buyers while holding the overall matching rate constant. Because of the second part of the above assumption, we can continue to write $m(\lambda) \equiv \phi(\lambda, \lambda, \sigma)$.

We first consider the optimality of complete separation. As the following proposition establishes, assumption 7 implies that if—for a given endowment of buyers B_1 and B_2 and a given buyer value dispersion x_2—complete separation is optimal for some σ^b, then it is also optimal for all σ^a with $\sigma^a < \sigma^b$. That is, the parameter range for which complete separation is optimal is shrinking with σ.

Proposition 11. *Under Assumption 7, the area in which complete separation is optimal is shrinking in* σ.

Proof. Because $m(\lambda)$ is independent of σ (Assumption 7), total surplus generated by complete separation is independent of σ. To see this, suppose that the planner allocates α sellers to the submarket of x_2 buyers and the remaining sellers to the submarket of x_1 buyers, then total surplus is $\alpha m(\frac{B_2}{\alpha})(1+(x_2-1)) + (1-\alpha)m(\frac{B_1}{1-\alpha})$, which is certainly independent of σ. Thus conditional on complete separation, the optimal α, α^*, is also independent of σ.

Next, consider a general allocation with L submarkets. When $\sigma = \sigma^b$, by assumption we have $\alpha^* m(\frac{B_2}{\alpha^*})(1+(x_2-1)) + (1-\alpha^*)m(\frac{B_1}{1-\alpha^*}) \geq \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell, \sigma^b)$, where $S(\mu^\ell, \lambda^\ell, \sigma^b)$ is given by equation (7) and now depends also on σ. Since for any μ and λ, $\phi(\mu, \lambda, \sigma^b) \geq \phi(\mu, \lambda, \sigma^a)$, we have $S(\mu, \lambda, \sigma^b) \geq S(\mu, \lambda, \sigma^a)$. Therefore, $\alpha^* m(\frac{B_2}{\alpha^*})(1+(x_2-1)) + (1-\alpha^*)m(\frac{B_1}{1-\alpha^*}) \geq \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell, \sigma^b) \geq \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell, \sigma^a)$. Thus complete separation is also optimal for $\sigma = \sigma^a$. □

Next, we consider the case of complete pooling. We are interested in the following question: if—for a given endowment of buyers B_1 and B_2 and a given buyer value dispersion x_2—pooling is optimal for some σ, then under what conditions will pooling continue to be optimal for $\sigma + \Delta\sigma$?

By Proposition 3, pooling is optimal at a given σ if and only if $\overline{S}'(B_1, \sigma) \geq 0$. If this inequality is strict, then by continuity with respect to σ, it continues to hold for $\sigma + \Delta$. Hence, the more complicated case is the one in which $\overline{S}'(B_1, \sigma) = 0$; pooling then continues to be optimal for $\sigma + \Delta\sigma$ if and only if $\overline{S}'(B_1, \sigma + \Delta\sigma) \geq 0$, which is equivalent to[3]

$$\frac{\partial T_1(\mu, \lambda, \sigma)}{\partial \sigma} \geq g'(R(B_2, B_1 + B_2, \sigma)) \frac{\partial R(\mu, \lambda, \sigma)}{\partial \sigma}. \tag{55}$$

By equations (8) and (10), we have

$$\frac{\partial T_1(\mu, \lambda, \sigma)}{\partial \sigma} = (x_2 - 1)\phi_{\lambda\sigma} \quad \text{and} \quad \frac{\partial R(\mu, \lambda, \sigma)}{\partial \sigma} = (x_2 - 1)\left(\phi_\sigma - \mu\phi_{\mu\sigma} - \lambda\phi_{\lambda\sigma}\right).$$

Moreover, by equation (32), we have $g'(R(B_2, B_1 + B_2, \sigma)) = -1/\Lambda(\mu, \lambda, \sigma)$, where Λ is defined in 35, and $1/\infty = 0$ by convention. As a result, we can rewrite (55) as

$$\frac{1}{\Lambda(\mu, \lambda, \sigma) - \lambda}\left(\phi_\sigma - \mu\phi_{\mu\sigma}\right) \geq -\phi_{\lambda\sigma}, \tag{56}$$

which leads to the following result regarding the parameter range in which complete pooling is optimal.

Proposition 12. *Under Assumption 1, 2, 3, 5, and 7, the area in which complete pooling is optimal is expanding with* σ *if and only if* (56) *holds for all* (μ, λ).

Proof. See above. □

Next we show that (56) holds for the geometrically truncated geometric meeting technology, so the result applies.

[3]Note that the function g is independent of σ because $m(\lambda)$ is independent of σ.

Since $\phi(\mu, \lambda)$ is given by equation (44), equation (34) which determines $\Lambda(\lambda z, \lambda, \sigma)$ can be rewritten as

$$\frac{(1+\lambda)(\lambda_0 - \lambda)}{2 + \lambda + \lambda_0} = \frac{\lambda z \sigma}{1-\sigma}$$

As a function of λ_0, the supremum of the left-hand side is $1+\lambda$. Thus if $\frac{\lambda z\sigma}{1-\sigma} < 1+\lambda$, the solution to the above equation is given by

$$\Lambda(\lambda z, \lambda, \sigma) = \frac{\lambda\left((1+\lambda)(1-\sigma) + \sigma z(1+\lambda)\right)}{(1+\lambda)(1-\sigma) - \lambda z \sigma}. \tag{57}$$

Otherwise $\Lambda(\lambda z, \lambda, \sigma) = \infty$. To verify(56), we rewrite the above equation as

$$\frac{1}{\Lambda(\lambda z, \lambda, \sigma) - \lambda} = \frac{(1+\lambda)(1-\sigma) - \lambda z \sigma}{2\sigma z \lambda(1+\lambda)}. \tag{58}$$

Note that when $\frac{\lambda z\sigma}{1-\sigma} > 1+\lambda$, the right-hand side of the above equation is still well-defined, and it is negative (an underestimate of the true value, which is zero in this case).

Next, by equation (44) direct computation gives

$$\phi_\sigma(\lambda z, \lambda, \sigma) - \lambda z \phi_{\mu\sigma}(\lambda z, \lambda, \sigma) = z^2\lambda^2 \frac{1 + (1+\sigma - z\sigma)\lambda}{(1+(1-\sigma(1-z))\lambda)^3} > 0$$
$$\phi_{\lambda\sigma}(\lambda z, \lambda, \sigma) = z\lambda \frac{1 + ((2-\sigma)z - (1-\sigma))\lambda}{(1+(1-\sigma(1-z))\lambda)^3}.$$

From the above equations we can see that $\phi_\sigma - \lambda z \phi_{\mu\sigma}$ is always strictly positive, but the sign of $\phi_{\lambda\sigma}$ is indeterminate. A sufficient condition for (56) is thus we plug the right-hand side of equation (58) into (56) irrespective of whether $\frac{\lambda z\sigma}{1-\sigma} > 1+\lambda$, which then gives

$$\frac{(\phi_\sigma - \mu\phi_{\mu\sigma})}{\Lambda(\lambda z, \lambda, \sigma) - \lambda} + \phi_{\lambda\sigma} = \lambda z \frac{1 + \sigma + (\lambda + \lambda\sigma(z-1))^2 + \lambda((2-\sigma)(1+z\sigma) + \sigma^2)}{2(\lambda+1)\sigma(1+(1-\sigma(1-z))\lambda)^3}$$

The right-hand side is always positive. We have thus proved (56) for this particular meeting technology.

C Market Equilibrium with N Buyer Types

In this section, we show that in the general model with N buyer types, no seller can do better in equilibrium than posting a second-price auction combined with either a reserve price or a meeting fee. The reserve price can be positive or negative, where the latter just means that the seller is willing to sell the good at a price below his valuation, which we normalized to 0. Similarly, the meeting fee can be positive, in which case it is paid by each buyer meeting the seller, or negative, in which case payments take place in the opposite direction. Finally, we establish that equilibrium is constrained efficient.

C.1 Incentive Compatibility and Payoffs

When a monopolistic seller offers a selling mechanism, incentive compatibility requires that buyers' expected utility is intimately connected with their trading probabilities (see Myerson, 1981; Riley and Samuelson, 1981). This logic can be extended to an environment with competing sellers.[4] In such

[4]See Peters (2013) for a similar treatment for an invariant meeting technology.

an environment, the expected payoff that a buyer receives from visiting a submarket is equal to what he would get at a monopolistic seller with a random number of buyers as in Levin and Smith (1994). However, buyers must also choose which submarket to visit and this depends on the posted mechanisms which in turn depends on the meeting technology.

In our analysis, it will sometimes be useful to consider buyers with a value x that is not in the set $\{x_1, \ldots, x_N\}$, who thus have measure zero. To do so, we define an extended version of the market utility function $U(x)$, which represents the highest expected payoff that a buyer with value x can achieve, such that $U_k \equiv U(x_k)$ for each k. Given any set of mechanisms posted by sellers, denote the set of mechanisms that buyers of type x visit by $\Omega^b(x)$, pick an arbitrary $\omega^b(x) \in \Omega^b(x)$ and denote by $p(x, \omega^b(x))$ the probability that a buyer of type x who visits a mechanism $\omega^b(x)$ trades. Of course, if buyers of type x choose to be inactive, then we set $\omega^b(x) = \emptyset$ and $p(x, \emptyset) = 0$. The following Lemma then establishes the properties of the market utility function. Its proof is closely related to the one in Myerson (1981).

Lemma 13. *Given any set of mechanisms posted by sellers, $p(x, \omega^b(x))$ is non-decreasing and the market utility function $U(x)$ is convex, satisfying*

$$U(x) = U(0) + \int_0^x p(z, \omega^b(z))dz.$$

Proof. The strategy of a buyer with value x is: (i) a probability distribution over the mechanisms to visit and inactivity and (ii) a value to report when the mechanism is not inactivity. Given the mechanisms posted by sellers, suppose that the set of mechanisms that a buyer with valuation x visits is $\Omega^b(x)$, and the probability that the buyer receives the object when visiting seller $\omega \in \Omega^b(x)$ and reporting x by $p(x, \omega)$, with a corresponding expected payment $t(x, \omega)$.

First, we select one element $\omega^b(z) \in \Omega^b(z)$ for each z. Then, by the incentive compatibility constraint (ICC), for any x, z,

$$U(x) \geq xp(z, \omega^b(z)) - t(z, \omega^b(z)), \tag{59}$$

i.e., buyers with valuation x are always better of following their equilibrium strategies than mimicking any other type z. Therefore,$U(x) = \max_{z\in[x_1,x_N]} xp(z, \omega^b(z)) - t(z, \omega^b(z))$. Hence, $U(x)$ is the supreme of a collection of affine functions and must therefore be convex.

Furthermore, we can rewrite equation (59) in the following way.

$$\begin{aligned} U(x) = xp(x, \omega^b(x)) - t(x, \omega^b(x)) &\geq xp(z, \omega^b(z)) - t(z, \omega^b(z)) \\ &= U(z) + p(z, \omega^b(z))(x - z). \end{aligned}$$

So, $p(x, \omega^b(x))$ is the slope of a supporting line for the convex function $U(x)$. Therefore, $p(x, \omega^b(x))$ is a non-decreasing function, and it equals the derivative of $U(x)$ almost everywhere. The latter then implies the integral representation of $U(x)$ in Lemma 13. □

As the supremum of a collection of convex functions (expected payoffs from individual submarkets), the market utility function $U(x)$ is always convex. Because of incentive compatibility, a higher winning probability is associated with a higher expected payoff.

Buyers only visit sellers who offer them their market utility and the sellers are residual claimants of the output. Competition forces sellers to post an efficient mechanism, i.e. a mechanism in which the buyer with the highest value trades if and only if his valuation exceeds that of the seller. In other words, efficient mechanisms are the cheapest way to offer buyers their market utility.

Consider a submarket with an efficient mechanism and a queue $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_N)$ where the lowest type of buyers visiting the submarket is ι. Consider a buyer with value x strictly between x_{k-1} and x_k

with $k \geq \iota + 1$. Since the posted mechanism is efficient, his winning probability is $\phi_\mu(\mu_k, \mu_1)$, which, by equation (6), is the probability that the buyer meets a seller and has the highest value among all buyers who arrived at the seller. As in a monopolistic auction, buyers' expected value is a summation (or with continuous types, an integral) over their winning probabilities. The expected value for the buyer is $V(x) = V_{k-1} + (x - x_{k-1})\phi_\mu(\mu_k, \mu_1)$. Since $V(x)$ must be continuous, the expected payoff for a buyer with valuation x_k visiting this submarket is $V_k = V_{k-1} + (x_k - x_{k-1})\phi_\mu(\mu_k, \mu_1)$, which then implies

$$V_k = V_\iota + \sum_{j=\iota+1}^{k} (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1). \tag{60}$$

The expected payoff for buyers in equation (60) is similar to the corresponding payoff in a monopolistic auction. Equation (60) also shows that $\phi_\mu(\mu_k, \mu_1)$ and $\phi_\mu(\mu_{k+1}, \mu_1)$ are subgradients at point x_k for the market utility function $U(x)$, since $V(x)$ lies below $U(x)$ and the slope for $V(x)$ with $x \in (x_{k-1}, x_k)$ is $\phi_\mu(\mu_k, \mu_1)$, and the slope for $V(x)$ with $x \in (x_k, x_{k+1})$ is $\phi_\mu(\mu_{k+1}, \mu_1)$. Two special cases are worth mentioning. Suppose that the lowest and the highest value of buyers who visit the submarket are $\underline{x}$ and $\overline{x}$, respectively. Then $Q_1(\mu_1) = \phi_\mu(\mu_1, \mu_1)$ and $1 - Q_0(\mu_1) = \phi_\mu(0, \mu_1)$ are subgradients at point $\underline{x}$ and $\overline{x}$, respectively. A buyer with value $x > \overline{x}$ will always trade as long as he successfully meets a seller, which happens with probability $1 - Q_0(\mu_1) = \phi_\mu(0, \mu_1)$.

Since the mechanism is assumed to be efficient, the expected seller value is given by

$$\begin{aligned}
\pi &= S(\boldsymbol{\mu}) - \sum_{k=\iota}^{N} (\mu_k - \mu_{k+1}) V_k \\
&= \sum_{k=\iota}^{N} (x_k - x_{k-1})\phi(\mu_k, \mu_1) - \sum_{k=\iota}^{N} (\mu_k - \mu_{k+1}) \left(V_\iota + \sum_{j=\iota+1}^{k} (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1) \right) \\
&= -\mu_1 V_\iota + \sum_{k=\iota}^{N} x_k(\phi(\mu_k, \mu_1) - \phi(\mu_{k+1}, \mu_1)) - \sum_{j=\iota+1}^{N} \mu_j (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1).
\end{aligned}$$

where in deriving the last equality we changed the order of summation. Rewriting the above equation yields

$$\pi = -\mu_1 V_\iota + \sum_{j=\iota+1}^{N+1} \left(x_{j-1} - \frac{\mu_j \phi_\mu(\mu_j, \mu_1)(x_j - x_{j-1})}{\phi(\mu_{j-1}, \mu_1) - \phi(\mu_j, \mu_1)} \right) \big(\phi(\mu_{j-1}, \mu_1) - \phi(\mu_j, \mu_1)\big). \tag{61}$$

To make the comparison with the classic auction literature more clear, we take the limit of the discrete buyer value distribution so that it converges to a continuous distribution F with density f. Then let $\lambda \equiv \mu_1$ and we have $\mu_j = \lambda(1 - F(x_j))$. Let $x_j = x$ and $x_{j-1} = x - \Delta x$, then the summand in equation (61) becomes

$$\left(x - \Delta x - \frac{\lambda(1 - F(x))\phi_\mu(\lambda(1 - F(x)), \lambda)\Delta x}{\phi(\lambda(1 - F(x - \Delta x)), \lambda) - \phi(\lambda(1 - F(x)), \lambda)} \right) \big(\phi(\lambda(1 - F(x - \Delta x)), \lambda) - \phi(\lambda(1 - F(x)), \lambda)\big).$$

Letting $\Delta x \to 0$, the first term becomes

$$x - \Delta x - \frac{\lambda(1 - F(x))\phi_\mu(\lambda(1 - F(x)), \lambda)\Delta x}{\phi_\mu(\lambda(1 - F(x)), \lambda)\lambda f(x)\Delta x} \to x - \frac{1 - F(x)}{f(x)},$$

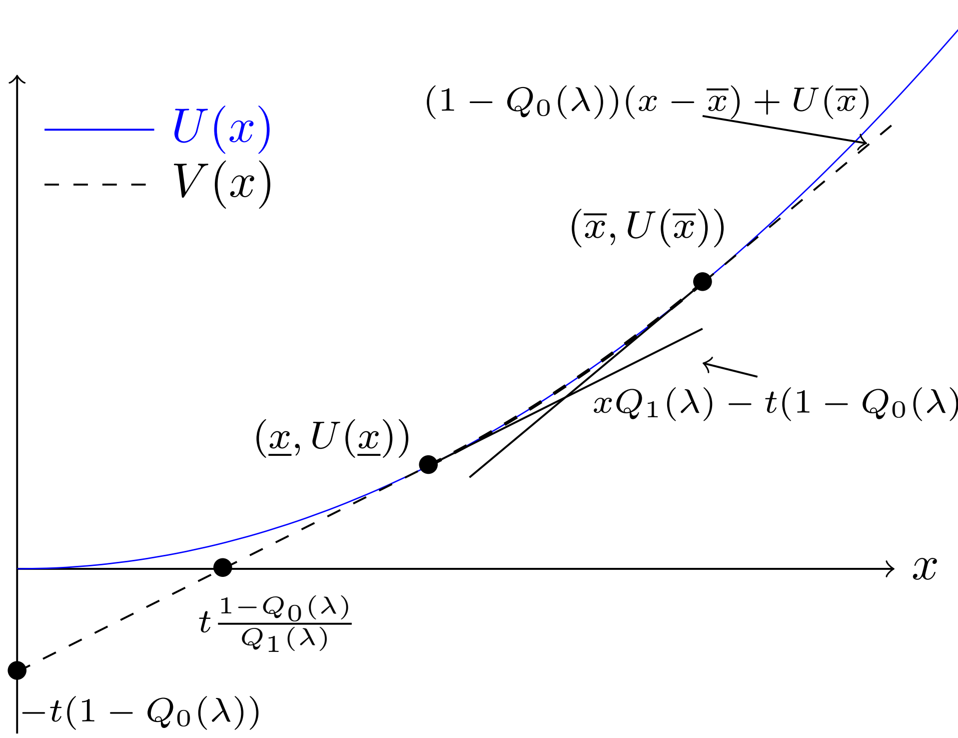


Figure 3: Supporting lines

and the second term is the measure of the distribution function $1 - \phi(\lambda(1 - F(x)), \lambda)$ between x and $x + \Delta x$. Therefore, we can rewrite equation (61) in the more familiar integral form

$$\pi = -\lambda V_\iota + \int_{x_\iota}^{x_N} (x - \frac{1 - F(x)}{f(x)}) d\left(1 - \phi(\lambda(1 - F(x)), \lambda)\right), \tag{62}$$

where $\lambda = \mu_1$.

In a standard auction with n bidders, a seller's expected payoff equals the virtual valuation function integrated against the distribution of the highest valuation, see Myerson (1981). Our setting is different in two ways: (i) because the buyer value distribution is discrete, the virtual value function takes a slightly more complicated form, and (ii) the distribution of the highest valuation of bidders depends on the meeting technology and is given by $1 - \phi(\mu_j, \mu_1)$, i.e., the probability that there are no buyers with valuations above x_j.

One may have expected that allowing for general meeting technologies would severely complicate the payoff functions in (competing) auction theory. We have shown here that our alternative representation ϕ avoids such complications. In particular, agents' expected payoffs retain the same structure but simply depend on transformations of ϕ.

C.2 Efficiency

Equivalence. To prove constrained efficiency of equilibrium, we show that even if sellers can buy queues directly in a hypothetical competitive market, they cannot do better than in the decentralized environment. In other words, the following two problems are equivalent for sellers.

1. Sellers' Relaxed Problem, in which there exists a hypothetical competitive market for queues, with the price for each buyer given by the market utility function. That is, sellers choose a queue $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_N)$ to maximize

$$\pi(\boldsymbol{\mu}) = \sum_{j=1}^{N}(x_j - x_{j-1})\phi(\mu_j, \mu_1) - \sum_{j=1}^{N}(\mu_j - \mu_{j+1})U_j, \tag{63}$$

where the first term is total surplus (22) and the second term is the price of the queue.

2. Sellers' Constrained Problem, in which sellers must post mechanisms to attract queues of buyers, as described in detail in Section 2. For any mechanism, the corresponding queue must be compatible with the market utility function, which means that it needs to satisfy equation (2). In this case, a seller's profit is again given by equation (63), but now queue length and queue composition depend on the posted mechanism.

In the relaxed problem, a seller will "buy" buyers with valuation x_k until their marginal contribution T_k to surplus is equal to their marginal cost U_k. Hence, if sellers can post a mechanism which delivers buyers their marginal contribution to surplus, then buyers' payoffs are equal to their market utility and the queue is compatible with the mechanism and the market utility function, as defined by equation (2). The following proposition establishes that auctions with an entry fee or a reserve price can achieve this.

Proposition 13. *Given that the market utility function is convex, any solution $\boldsymbol{\mu}$ to the sellers' relaxed problem is compatible with an auction with an entry fee in the sellers' constrained problem, where the fee is given by*

$$t = -\frac{\sum_{j=1}^{N}(x_j - x_{j-1})\phi_\lambda(\mu_j, \mu_1)}{1 - Q_0(\mu_1)}. \tag{64}$$

It is also compatible with an auction with a reserve price in the sellers' constrained problem, where the reserve price is given by

$$r = -\frac{\sum_{j=1}^{N}(x_j - x_{j-1})\phi_\lambda(\mu_j, \mu_1)}{Q_1(\mu_1)}. \tag{65}$$

Proof. In their relaxed problem, sellers select a queue $\boldsymbol{\mu}$ directly in a hypothetical competitive market. The expected payoff for a seller in this market is the difference between the surplus that he creates and the price of the queue. Suppose that a queue $\boldsymbol{\mu}$ solves sellers' relaxed problem. If queue $\boldsymbol{\mu}$ contains buyers of value x_k, then $T_k(\boldsymbol{\mu}) = U_k$, where $T_k(\boldsymbol{\mu})$ is given by equation (23); if queue $\boldsymbol{\mu}$ does not contain buyers of value x_k ($\mu_k = \mu_{k+1}$), then $T_k(\boldsymbol{\mu}) \leq U_k$.

Note that when a seller posts a second-price auction with entry fee and t is given by equation (64), note that $V_\iota = T_\iota(\boldsymbol{\mu})$, where ι is the lowest buyer value in queue $\boldsymbol{\mu}$. The important observation is that by equation (23) and (60), for all $k \geq 1$ we have $V_k = T_k(\boldsymbol{\mu})$, where V_k is the expected payoff of buyers with value x_k from the submarket and is given by equation (60). Thus $\boldsymbol{\mu}$ is also compatible with a second-price auction with entry fee t in the sellers' constrained problem.

The case with the reserve price is similar except for one difference. When a seller posts a second-price auction with entry fee t given by equation (64), $V_k = T_k(\boldsymbol{\mu})$ for $k \geq 1$. But when a seller posts a second-price auction with reserve price r given by equation (65), $V_\iota = T_\iota(\boldsymbol{\mu})$ where ι is the lowest buyer value in queue $\boldsymbol{\mu}$. For $k > \iota$, V_k is again given by equation (60) and we have $V_k = T_k(\boldsymbol{\mu})$. For $k < \iota$, things are slightly more complicated: For $r < x_k < x_\iota$, $V_k = Q_1(\mu_1)(x_k - r) = V_\iota - \phi_\mu(\mu_1, \mu_1)(x_\iota - x_k)$, which implies $V_k = T_k(\boldsymbol{\mu})$. For $x_k < r$, $V_k = 0$ and $T_k(\boldsymbol{\mu}) < 0$. In this case buyers with value x_k will not visit the submarket. Thus queue $\boldsymbol{\mu}$ is compatible with a second-price auction with reserve price r in the sellers' constrained problem. □

Efficiency. Proposition 13 is an important step towards proving efficiency of the market equilibrium for general meeting technologies, but there is one remaining issue: for a given auction with a reserve price or entry fee, there might be multiple queues compatible with the market utility function. Therefore, even if a solution to the sellers' relaxed problem is compatible with an auction with reserve price or entry fee, it is not clear that sellers will expect that solution to be the realized queue. Most of the literature resolves this issue by assuming that sellers are optimistic: a (deviating) seller expects that he can coordinate buyers in such a way that the solution to the sellers' relaxed problem becomes the realized queue.[5] Since this assumption is somewhat arbitrary, we show in the next subsection that we can relax it under some mild restrictions on the meeting technology. However, if we—for the moment—follow the standard approach, then by Proposition 13, a seller's relaxed and constrained problem are equivalent in the sense that they achieve the same outcome. That is, the directed search equilibrium is equivalent to a competitive market equilibrium for queues, which also coincides with the socially efficient planner's allocation.

Proposition 14. *If sellers are optimistic, the directed search equilibrium is constrained efficient for any meeting technology.*

Proof. The sellers' relaxed problem boils down to a competitive market for buyer types. Therefore, the first welfare theorem applies and the equilibrium is efficient. Since the sellers' constrained problem is equivalent to the sellers' relaxed problem, the directed search equilibrium is also efficient. □

Hence, we have shown that despite the potential presence of spillovers in the meeting process, business stealing externalities and agency costs, the competing mechanisms problem reduces to one where sellers can buy queues in a competitive market. This result, of course, requires a sufficiently large contract space. If it is not possible for sellers to either commit to a reserve price above their valuation or charge fees, the decentralized equilibrium will only be efficient for invariant meeting technologies (i.e. $\phi_\lambda = 0$). If $\phi_\lambda < 0$ (resp. > 0), buyers impose negative (resp. positive) externalities on other meetings and will receive more (resp. less) than their marginal social contribution.[6]

C.3 Uniqueness of Beliefs

Without the optimism assumption, payoff equivalence of all equilibria could break down if multiple queues are compatible with market utility. In this subsection, we show that such a scenario is rather special in the sense that—under mild restrictions on the meeting technology—the solution to the market utility condition is in fact unique, rendering the optimism assumption redundant.

Uniqueness. The following proposition then presents our result regarding uniqueness of the beliefs for a seller posting a second-price auction with a reserve price.

Proposition 15. *Under assumptions 1 and 2, for each seller posting a second-price auction with a reserve price r, there is a unique queue $\boldsymbol{\mu}$ compatible with the market utility function. Furthermore, for two sellers posting reserve prices r^a and r^b, $r^a < r^b$ if and only if $\mu_1^a > \mu_1^b$.*

[5] See, for example, Eeckhout and Kircher (2010a,b).

[6] With free entry of sellers, the buyer-seller ratio would be too high (resp. too low) in this case.

Proof. Our proof consists of two parts: i) $r^a < r^b \Leftrightarrow \mu_1^a > \mu_1^b$, and ii) queue length μ_1 determines the whole queue $\boldsymbol{\mu}$ uniquely. We first prove the second part.

As we showed in the proof of Proposition 10, a given queue length determines the lowest buyer type $\underline{x}$ and the highest buyer type $\overline{x}$ uniquely. If $\underline{x} < x_k \le \overline{x}$, μ_k is uniquely determined by $U_k = U_{k-1} + (x_k - x_{k-1})\phi_\mu(\mu_k, \mu_1)$. Thus queue length determines the whole queue uniquely.

Next we move to the first part. Again as we showed in the proof of Proposition 10, queue length μ_1 determines $\underline{x}$ uniquely, and $\mu_1^a > \mu_1^b$ implies that $\underline{x}^a \le \underline{x}^b$. Also note that in some submarket ℓ, $U(\underline{x}^\ell) = Q_1(\mu_1^\ell)(\underline{x}^\ell - r^\ell)$, which implies that the reserve price can be written as $r^\ell = \underline{x}^\ell - U(\underline{x}^\ell)/Q_1(\mu_1^\ell)$. Combining the above two facts shows that queue length μ_1 completely determines the reserve price r. Next we show that $r^a < r^b$.

If $\underline{x}^a = \underline{x}^b$, then $r^a = \underline{x}^a - U(\underline{x}^a)/Q_1(\mu_1^a) < \underline{x}^b - U(\underline{x}^b)/Q_1(\mu_1^b) = r^b$ because $Q_1(\mu_1^a) < Q_1(\mu_1^b)$ (see Assumption 2).

Next consider the case $\underline{x}^a < \underline{x}^b$. Since $\phi_\mu(\mu_1^i, \mu_1^i)$ $(Q_1(\mu_1^i))$ is a subgradient at $\underline{x}^i$, we have

$$U(\underline{x}^a) > U(\underline{x}^b) + Q_1(\mu_1^b)(\underline{x}^a - \underline{x}^b), \tag{66}$$

which implies that

$$\underline{x}^b - \underline{x}^a > \frac{U(\underline{x}^b)}{Q_1(\mu_1^b)} - \frac{U(\underline{x}^a)}{Q_1(\mu_1^b)} \ge \frac{U(\underline{x}^b)}{Q_1(\mu_1^b)} - \frac{U(\underline{x}^a)}{Q_1(\mu_1^a)},$$

where the second inequality follows from $Q_1(\lambda^a) < Q_1(\lambda^b)$. Recall that $r^i = \underline{x}^i - U(\underline{x}^i)/Q_1(\mu_1^i)$ for for $i = a$ or b. We then have $r^a < r^b$. $\square$

The above result can be easily illustrated graphically. In Figure 4, the intersection between the supporting line with slope $Q_1(\mu_1)$ at $\underline{x}$ and the x-axis is simply r. Consider two different queues a and b. If queue a is longer ($\mu_1^a > \mu_1^b$), then we have $Q_1(\mu_1^a) < Q_1(\mu_1^b)$ and hence $\underline{x}^a \le \underline{x}^b$. Since the reserve price r is the intersection point of the supporting line at $\underline{x}$ with slope $Q_1(\lambda)$ and the x-axis, from Figure 4 we can see that $\underline{x}^a \le \underline{x}^b$ implies that $r^a < r^b$.[7] Of course, this logic can also be reversed so that a lower reserve price implies a longer queue.

Things are slightly more complicated when sellers post a second-price auction with an entry fee. Below, we introduce one weak additional restriction on the meeting technology, which is sufficient to guarantee that there exists a monotonic relation between meeting fees and queue lengths. This implies that there exists a unique queue that is compatible with the market utility function when sellers post an auction with an entry fee.

Assumption 8. $(1 - Q_0(\lambda))/Q_1(\lambda)$ *is weakly increasing in* λ.

If we rewrite $(1 - Q_0(\lambda))/Q_1(\lambda)$ as $1 + \sum_{k=2}^{\infty} Q_k(\lambda)/Q_1(\lambda)$, then this assumption states that with a higher buyer-seller ratio, it is relatively more likely that a buyer will meet competitors in an auction rather than being alone.

Proposition 16. *Under assumptions 1, 2, and 8, for each seller posting an auction with entry fee* t*, there is a unique queue* $\boldsymbol{\mu}$ *compatible with the market utility function. Furthermore, for two sellers posting entry fees* t^a *and* t^b*,* $t^a < t^b$ *if and only if* $\mu_1^a > \mu_1^b$*.*

Proof. Assume that $\mu_1^a > \mu_1^b$, which then implies $\underline{x}^a \le \underline{x}^b$. We distinguish two cases $t^b < 0$ and $t^b \ge 0$. First consider the case $t^b < 0$ (entry subsidy). We prove the claim by contradiction. Suppose that

[7] The expected utility of a buyer with valuation $r^a < \underline{x}$ of visiting segment a is $U(r^a) = (r^a - r^a)Q_1(\mu_1^a) = 0$, this implies that the support line intersects the horizontal axis at $x = r^a$. In equilibrium type $x = r^a$ would not visit segment a, since it can obtain a positive expected utility in another submarket.

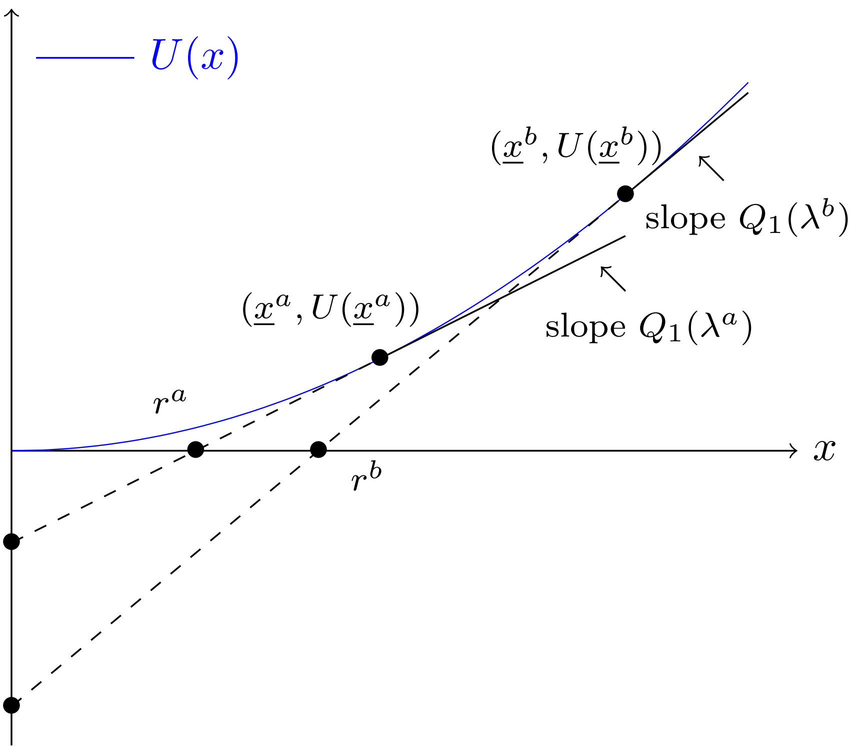


Figure 4: Relation between reserve prices and queue length

$t^a \geq t^b$. Then buyers with value $\underline{x}^a$ will receive a strictly higher payoff by switching to queue b. To see this, the expected payoff for buyers with value $\underline{x}^a$ after switching to queue b is $Q_1(\mu_1^b)\underline{x}^a - (1-Q_0(\mu_1^b))t^b$. Note that

$$Q_1(\mu_1^b)\underline{x}^a - (1 - Q_0(\mu_1^b))t^b > Q_1(\mu_1^a)\underline{x}^a - (1 - Q_0(\mu_1^b))t^b \geq Q_1(\mu_1^a)\underline{x}^a - (1 - Q_0(\mu_1^a))t^b$$
$$\geq Q_1(\mu_1^a)\underline{x}^a - (1 - Q_0(\mu_1^a))t^a = U(\underline{x}^a)$$

where the first inequality is because $Q_1(\lambda)$ is strictly decreasing, the second inequality is because $1 - Q_0(\lambda)$ is strictly decreasing and $t^b \leq 0$, and the final inequality follows from the assumption that $t^a \geq t^b$. Thus we have reached a contradiction, and it must be true that $t^a < t^b$.

Next we consider the case $t^b > 0$. Note that there is a simple relation between the entry fee and the reserve price for a given queue: $Q_1(\mu_1)r = (1 - Q_0(\mu_1))t$. By Proposition 15, we then have $\mu_1^a > \mu_1^b \Leftrightarrow \frac{1-Q_0(\mu_1^a)}{Q_1(\mu_1^a)}t^a < \frac{1-Q_0(\mu_1^b)}{Q_1(\mu_1^b)}t^b$, which implies that

$$t^a < \frac{Q_1(\mu_1^a)}{1 - Q_0(\mu_1^a)} \frac{1 - Q_0(\mu_1^b)}{Q_1(\mu_1^b)} t^b \leq t^b.$$

where the second inequality is because of assumption 8 and $t^b > 0$. Again we have $t^a < t^b$.

As in the proof of Proposition 15, a given queue length determines the queue completely. Therefore, there exists at most one queue compatible with an entry fee t and $t^a < t^b \Leftrightarrow \mu_1^a > \mu_1^b$. □

The intuition behind Proposition 16 is similar to that of Proposition 15 and readily follows from the correspondence between the reserve price and entry fee: $t = rQ_1/(1-Q_0)$. Again, consider two different queues a and b. We have shown in Proposition 15 that $\mu_1^a > \mu_1^b \Leftrightarrow r^a < r^b$. Under Assumption 8, the two inequalities jointly lead to $t^a < t^b$.

Hence, we have established that under mild restrictions on the meeting technology, there exists only one queue which is compatible with market utility when sellers post an auction with a reserve price or entry fee. Consequently, the assumption that sellers are optimistic is redundant for a large class of meeting technologies.

References

Eeckhout, J. and Kircher, P. (2010a). Sorting and decentralized price competition. Econometrica, 78:539–574.

Eeckhout, J. and Kircher, P. (2010b). Sorting vs screening - search frictions and competing mechanisms. Journal of Economic Theory, 145:1354–1385.

Levin, D. and Smith, J. (1994). Equilibrium in auctions with entry. The American Economic Review, pages 585–599.

Myerson, R. B. (1981). Optimal auction design. Mathematics of Operations Research, 6(1):58–73.

Peters, M. (2013). Competing mechanisms. In Vulkan, N., Roth, A. E., and Neeman, Z., editors, The Handbook of Market Design, pages 609–628. Oxford University Press.

Riley, J. G. and Samuelson, W. F. (1981). Optimal auctions. The American Economic Review, 71(3):pp. 381–392.