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On the sample path properties of mixed Poisson processes

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ABSTRACT

The mixed Poisson process has been widely used in financial engineering for modeling arrival of events that cluster in time, as it has strictly stationary and positively correlated increments. However, we show that, surprisingly, the sample autocovariance and autocorrelation of the increments of a mixed Poisson process converge to zero almost surely as the sample size goes to infinity. Consequently, the sample autocovariance or autocorrelation cannot be used in the method of moments for parameter estimation of mixed Poisson processes.

1. Introduction

The mixed Poisson process is a generalization of the homogeneous Poisson process for modeling the occurrence of events. In contrast to the homogeneous Poisson process that has independent increments, the mixed Poisson process has the distinctive feature that the correlation between its increments is positive. Roughly speaking, if the number of events occurring in the current period is large, then the number of events that will occur in the next period will also tend to be large. Hence, the arrival of events under the mixed Poisson process has the feature of *clustering in time*, which has been empirically observed in various fields such as finance and insurance.

The mixed Poisson process has been widely used in the literature of financial engineering and insurance. The CreditRisk+ system [19] uses a mixed Poisson process with the Gamma distribution as the mixing distribution for the modeling of arrival of default events. [6,7] develop an importance sampling method for evaluating the loss distribution under a mixed Poisson model of portfolio credit risk. [4] provides a review of jump processes such as mixed Poisson processes and doubly stochastic Poisson processes and their applications in credit risk modeling. [12] proposes a model for pricing portfolio credit derivatives in which a mixed Poisson process is used as a market factor that can introduce clustering shocks to affect the default probabilities of firms. [10] proposes a mixed Poisson credit risk model in which loss-given-default is dependent on probability of default. [2] provides comprehensive discussion on credit risk models including the CreditRisk+ and its extensions. The mixed Poisson process has also been widely used for modeling the arrival of accidents, sickness, and insurance claims in the actuarial science literature (see, e.g., [13,20]).

In this paper, we study the asymptotics of the sample autocovariance and autocorrelation of the increments of a mixed Poisson process. Contrasting the homogeneous Poisson process, the increments of a mixed Poisson process are strictly stationary but have positive autocorrelation. Hence, one may expect that the asymptotic distribution of sample autocorrelation of the increments of a mixed Poisson process has a mean equal to the true positive autocorrelation (as the sample size goes to infinity), which may hold for a weakly stationary time series under some general conditions. For example, [3, Chap. 7] shows that the asymptotic mean of sample autocorrelation is equal to the population autocorrelation for weakly stationary ARMA(p, q) time series and for weakly stationary time series $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, where $\{Z_t\}$ is a sequence of i.i.d. random variables with mean zero and finite variance. See also [5, Chap. 6].

The main result of this paper is that, surprisingly, the sample autocovariance and autocorrelation of the increments of a mixed Poisson process converge to zero almost surely. This provides an interesting counterexample showing that even for a strictly stationary time series, the sample autocorrelation may not have an asymptotic mean that is equal to the population autocorrelation.

The main result of the paper also implies that the sample autocovariance or autocorrelation cannot be used in the method of moments for parameter estimation of a mixed Poisson process. The method of moments based on autocovariance is a commonly used method for parameter estimation for weakly stationary time series partly due to the simplicity of the method. For example, the parameters of AR(p) model can be consistently estimated by

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the Yule–Walker equation, which is based on equating population autocorrelation with sample autocorrelation (see, e.g., [3, Chap. 8.1]). Although we have not noticed any existing literature that use the method of moments for the estimation of parameters for mixed Poisson processes, our result provides a precaution against the use of such a method for mixed Poisson processes.

The paper also complements the study of the properties of mixed Poisson process in the literature of point processes and applied probability. [1] discusses the goodness-of-fit test of whether a point process can be adequately modeled as a mixed Poisson process and in particular a homogeneous Poisson process based on the characterization of a mixed Poisson process. [11] shows that the property that normalized event occurrence times are the order statistics of independent uniform random variables on (0, 1) characterizes the mixed Poisson processes within the class of general point processes. The characterization of mixed Poisson processes within the class of birth processes, stationary point processes, and general point processes are also studied in [1,9,13,14,17], among others; see Chapter 6 of [8] for comprehensive discussion and summary. When the mixing distribution of the mixed Poisson process has a gamma distribution, the mixed Poisson process is called a Pólya–Lundberg process (or Pólya process); see [16] for a comprehensive discussion. [15] provides a martingale characterization of Pólya–Lundberg processes within the class of mixed Poisson processes.

The remainder of the paper is organized as follows. In Section 2, we briefly review the properties of the increments of mixed Poisson processes. In Section 3, we prove the main result of this paper. Section 4 concludes.

2. The increments of mixed Poisson processes

We recall the definition of the mixed Poisson process (see, e.g., [18, Section 8.5.1]). A counting process $\{N(t), t \geq 0\}$ is called a mixed Poisson process if there exists a positive random variable Λ with distribution function $F_\Lambda(\cdot)$ such that for any n, any sequence $\{k_i \mid i = 1, 2, \ldots, n\}$ of nonnegative integers, and any sequence $0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n$,

$$P(N(b_i) - N(a_i) = k_i; i = 1, 2, \ldots, n) = \int_0^\infty \prod_{i=1}^n \frac{(\lambda(b_i - a_i))^{k_i}}{k_i!} e^{-\lambda(b_i - a_i)}\, dF_\Lambda(\lambda). \quad (1)$$

If $\{N(t), t \geq 0\}$ is a mixed Poisson process, then for any fixed t, the marginal distribution of $N(t)$ is a mixed Poisson distribution. Comparing with Poisson distribution, mixed Poisson distribution is overspersed in the sense that the variance of $N(t)$ is larger than $E[N(t)]$. A mixed Poisson process is a non-homogeneous pure birth process whose birth rate function is determined by the mixing distribution (see, e.g., [18, Theorem 8.5.1]). The interoccurrence times of a mixed Poisson process are identically distributed but not independent.

The easiest way to construct a mixed Poisson process is through stochastic time change of a Poisson process. More precisely, let Λ be a random variable with distribution function $F_\Lambda(\cdot)$, $\{M(t), t \geq 0\}$ be a Poisson process with intensity 1, and let Λ and $\{M(t), t \geq 0\}$ be independent. Define the process $\{N(t), t \geq 0\}$ as follows:

$$N(t) := M(\Lambda \cdot t), t \geq 0. \quad (2)$$

Then, the process $\{N(t), t \geq 0\}$ defined in Eq. (2) is a mixed Poisson process (see, e.g., Definition 4.3 in [8, p. 66]).

Let $\delta > 0$ be a constant and consider the increments of the mixed Poisson processes:

$$m_t := N(t\delta) - N((t-1)\delta), t = 1, 2, \ldots. \quad (3)$$

It is clear from (1) that the time series m_t is strictly stationary but not independent. In fact, the autocovariance (ACV) and autocorrelation (ACR) functions of m_t at any lag are positive, as is shown in the following proposition.

Proposition 2.1. *$\{m_t \mid t = 1, 2, \ldots\}$ defined in (3) is a strictly stationary time series with the following autocovariance (ACV) and autocorrelation (ACR) functions:*

$$c_k = Cov(m_t, m_{t-k}) = \delta^2 Var(\Lambda) > 0, \forall k > 0, \quad (4)$$

$$\rho_k = \frac{Cov(m_t, m_{t-k})}{Var(m_t)} = \frac{\delta^2 Var(\Lambda)}{\delta E[\Lambda] + \delta^2 Var(\Lambda)} > 0, \forall k > 0. \quad (5)$$

Proof. By direct computation. □

In particular, if $N(t)$ is a Pólya–Lundberg process with the Gamma distribution with shape parameter α and scale parameter β as the mixing distribution, then the ACV and ACR of $\{m_t\}$ are given by

$$c_k = Cov(m_t, m_{t-k}) = \delta^2 \alpha \beta^2, \rho_k = \frac{Cov(m_t, m_{t-k})}{Var(m_t)} = \frac{\delta\beta}{1 + \delta\beta}, \forall k > 0.$$

3. Main result

The main result of this paper is that the sample ACV and sample ACR at any lag of $\{m_t\}$ converge to zero almost surely, although the population ACV and population ACR of $\{m_t\}$ are strictly positive. As a result, the sample ACV or sample ACR cannot be used in the method of moments for parameter estimation of mixed Poisson processes. The sample ACV and sample ACR at lag k of $\{m_t\}$ are defined as follows:

$$\bar{m}_T := \frac{1}{T} \sum_{t=1}^T m_t = \frac{1}{T} N(T\delta) \quad (6)$$

$$\hat{c}_{k,T} := \frac{\sum_{t=k+1}^T (m_t - \bar{m}_T)(m_{t-k} - \bar{m}_T)}{T}. \quad (7)$$

$$\hat{\rho}_{k,T} := \frac{\sum_{t=k+1}^T (m_t - \bar{m}_T)(m_{t-k} - \bar{m}_T)}{\sum_{t=1}^T (m_t - \bar{m}_T)^2}. \quad (8)$$

First, we show that the sample ACV at any lag of the increments of a homogeneous Poisson process converge to zero almost surely.

Lemma 3.1. *Let $M(t)$ be a homogeneous Poisson process with intensity 1. Let $\lambda > 0$ be a constant and $k > 0$ be a constant integer. Define*

$$m_{t,\lambda} := M(t\delta\lambda) - M((t-1)\delta\lambda), \quad (9)$$

$$\bar{m}_{T,\lambda} := \frac{1}{T} \sum_{t=1}^T m_{t,\lambda}, \quad (10)$$

$$\hat{c}_{k,T}(\lambda) := \frac{\sum_{t=k+1}^T [m_{t,\lambda} - \bar{m}_{T,\lambda}][m_{t-k,\lambda} - \bar{m}_{T,\lambda}]}{T}. \quad (11)$$

Then, it holds that

$$\lim_{T \to \infty} \hat{c}_{k,T}(\lambda) = 0, \ a.s. \quad (12)$$

Proof. Omitted due to limited space. Details can be provided upon request. □

Second, we consider the mixed Poisson process $N(t)$ defined in (2). For this process $N(t)$, an intuitive but unrigorous argument to show $P(\lim_{T\to\infty}\hat{c}_{k,T}(\lambda)=0)=1$ is as follows: for any constant $\lambda>0$, we have

$$\begin{aligned}&P(\lim_{T\to\infty}\hat{c}_{k,T}=0 \mid \Lambda=\lambda)\\=&P(\lim_{T\to\infty}\hat{c}_{k,T}(\lambda)=0),\ (\hat{c}_{k,T}(\lambda) \text{ is defined in (11)})\\=&0.\ (\text{by Lemma 3.1}).\end{aligned} \tag{13}$$

The argument is unrigorous because the event $\{\Lambda=\lambda\}$ may have zero probability for continuous random variable Λ. The rigorous proof actually takes much more efforts. In the following, we will state the main result of the paper and provide a rigorous proof.

Theorem 3.1. *(i) Let $N(t)$ be a mixed Poisson process defined in (2). Let $k>0$ be a constant integer. Let m_t, $\hat{c}_{k,T}$, and $\hat{\rho}_{k,T}$ be defined in (3), (7), and (8) respectively. Then, it holds that*

$$\lim_{T\to\infty}\hat{c}_{k,T}=0,\ a.s. \tag{14}$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}m_t^2=\delta\Lambda+\delta^2\Lambda^2,\ a.s. \tag{15}$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(m_t-\bar{m}_T)^2=\delta^2\Lambda,\ a.s. \tag{16}$$

$$\lim_{T\to\infty}\hat{\rho}_{k,T}=0,\ a.s. \tag{17}$$

(ii) The statements (14) and (17) *hold for any mixed Poisson process which may not have the representation of (2).*

Proof. We use $I\{A\}$ to denote the indicator random variable on the event A.

(i) Let $N(t)$ be defined in (2). In the following, we will prove that the four statements (14)–(17) hold for $N(t)$.

(ii) We first prove (14). For each integer $n>0$, define the discrete random variable Λ_n as

$$\Lambda_n(\omega):=\begin{cases}\frac{s}{2^n}, & \Lambda(\omega)\in(\frac{s-1}{2^n},\frac{s}{2^n}],\ 1\le s\le 2^n\cdot n,\ \text{and } s\in Z^+,\\ n, & \Lambda(\omega)>n.\end{cases}$$

By definition, for $n\ge\Lambda(\omega)$, $\Lambda_n(\omega)\ge\Lambda(\omega)$.

First, we will show that for any fixed $\lambda\in(0,\infty)$, $\Delta\lambda\in(0,\infty)$, and $\omega\in\Omega$, there exists $0<G(\omega)<\infty$, such that the following three statements hold:

$$I\{\Lambda_n(\omega)<\lambda\}=I\{\Lambda(\omega)<\lambda\},\ \forall n>G(\omega), \tag{18}$$

$$\begin{aligned}0\le\ &\Lambda_n(\omega)I\{\lambda\le\Lambda_n(\omega)<\lambda+\Delta\lambda\}\\&-\Lambda(\omega)I\{\lambda\le\Lambda(\omega)<\lambda+\Delta\lambda\}<\frac{1}{2^n},\ \forall n>G(\omega),\end{aligned} \tag{19}$$

$$I\{\Lambda_n(\omega)\ge\lambda+\Delta\lambda\}=I\{\Lambda(\omega)\ge\lambda+\Delta\lambda\},\ \forall n>G(\omega). \tag{20}$$

Indeed, consider the following three cases:

1. If $\Lambda(\omega)\ge\lambda+\Delta\lambda$, then define $G(\omega):=\Lambda(\omega)$. Then, for $\forall n>G(\omega)$, $\Lambda_n(\omega)\ge\Lambda(\omega)\ge\lambda+\Delta\lambda$, which implies (18), (19), and (20) hold.
2. If $\lambda\le\Lambda(\omega)<\lambda+\Delta\lambda$, then define $G(\omega):=\max\{\lambda+\Delta\lambda,\log_2\frac{1}{\lambda+\Delta\lambda-\Lambda(\omega)}\}$. Then, for $\forall n>G(\omega)$, $\Lambda_n(\omega)\ge\Lambda(\omega)\ge\lambda$. In addition, since $\Lambda(\omega)<\lambda+\Delta\lambda<n$, it follows that $\Lambda_n(\omega)<\Lambda(\omega)+\frac{1}{2^n}$, which in combination with $n>\log_2\frac{1}{\lambda+\Delta\lambda-\Lambda(\omega)}$ implies that $\Lambda_n(\omega)<\Lambda(\omega)+\frac{1}{2^n}<\lambda+\Delta\lambda$. Therefore, $\lambda\le\Lambda_n(\omega)<\lambda+\Delta\lambda$, for $\forall n>G(\omega)$, which implies that (18), (19), and (20) hold.
3. If $\Lambda(\omega)<\lambda$, then define $G(\omega):=\max\{\Lambda(\omega),\log_2\frac{1}{\lambda-\Lambda(\omega)}\}$. Then, for $\forall n>G(\omega)$, $\Lambda_n(\omega)<\Lambda(\omega)+\frac{1}{2^n}<\lambda$, which implies that (18), (19), and (20) hold.

And then, by (19),

$$\begin{aligned}&\lim_{n\to\infty}\Lambda_n(\omega)I\{\lambda\le\Lambda_n(\omega)<\lambda+\Delta\lambda\}\\&=\Lambda(\omega)I\{\lambda\le\Lambda(\omega)<\lambda+\Delta\lambda\},\ \text{for any } \omega.\end{aligned} \tag{21}$$

For any fixed ω and $t>0$, since the sample paths of the Poisson process $M(t,\omega)$ are right-continuous step functions and (18), (19), and (20) hold, there exists $G_t(\omega)\in(0,\infty)$ such that

$$\begin{aligned}&M(t\delta\Lambda_n(\omega),\omega)I\{\lambda\le\Lambda_n(\omega)<\lambda+\Delta\lambda\}\\=&M(t\delta\Lambda(\omega),\omega)I\{\lambda\le\Lambda(\omega)<\lambda+\Delta\lambda\},\ \forall n>G_t(\omega).\end{aligned}$$

Therefore, for any fixed ω, it holds that

$$\begin{aligned}&M(t\delta\Lambda_n(\omega),\omega)I\{\lambda\le\Lambda_n(\omega)<\lambda+\Delta\lambda\}\\=&M(t\delta\Lambda(\omega),\omega)I\{\lambda\le\Lambda(\omega)<\lambda+\Delta\lambda\},\\&t=1,\ldots,T,\ \forall n>H_T(\omega),\end{aligned} \tag{22}$$

where $H_T(\omega):=\max\{G_1(\omega),G_2(\omega),\ldots,G_T(\omega)\}$.

Let $\hat{c}_{k,T}(\lambda)$ be defined in (11). We will show that for any integers $L<U$,

$$P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\}\mid\Lambda)=s_{L,U}(\Lambda), \tag{23}$$

where

$$s_{L,U}(\lambda):=P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\lambda)\in(a,b]\}).$$

Indeed, for any fixed ω and any sequence $\{\lambda_n\}\subset\mathbb{R}$ such that $\lambda_n\downarrow\lambda$ as $n\to\infty$, it follows from the right continuity of $\{M(t,\omega)\}$ that there exists $K_T(\omega)<\infty$ such that for $\forall n>K_T(\omega)$, $\hat{c}_{k,T}(\lambda_n)=\hat{c}_{k,T}(\lambda)$. Therefore,

$$\lim_{n\to\infty}I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\lambda_n)\in(a,b]\}\}=I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\lambda)\in(a,b]\}\},\ \text{a.s.}$$

And then, it follows from dominated convergence theorem that

$$\lim_{n\to\infty}P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\lambda_n)\in(a,b]\})=P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\lambda)\in(a,b]\}),$$

or, equivalently,

$$\lim_{n\to\infty}s_{L,U}(\lambda_n)=s_{L,U}(\lambda),\ \text{if } \lambda_n\downarrow\lambda \text{ as } n\to\infty. \tag{24}$$

Define

$$\begin{aligned}m_{t,n}(\omega):=&M(t\delta\Lambda_n(\omega),\omega)-M((t-1)\delta\Lambda_n(\omega),\omega),\\\bar{m}_{T,n}(\omega):=&\frac{1}{T}\sum_{t=1}^{T}m_{t,n}(\omega),\\\hat{c}_{k,T}(\Lambda_n(\omega),\omega):=&\frac{\sum_{t=k+1}^{T}[m_{t,n}(\omega)-\bar{m}_{T,n}(\omega)][m_{t-k,n}(\omega)-\bar{m}_{T,n}(\omega)]}{T}.\end{aligned}$$

Since $\hat{c}_{k,T}(\Lambda_n(\omega),\omega)$ is a function of $M(\delta\Lambda_n(\omega),\omega)$, $M(2\delta\Lambda_n(\omega),\omega)$, $\ldots$, $M(T\delta\Lambda_n(\omega),\omega)$, it follows from (22) that for any fixed integers $L<U$ and for any $(a,b]\subset\mathbb{R}$,

$$\begin{aligned}&I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda_n)\in(a,b]\}\}I\{\lambda\le\Lambda_n<\lambda+\Delta\lambda\}\\=&I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\}\}I\{\lambda\le\Lambda<\lambda+\Delta\lambda\},\ \forall n>H_U(\omega),\end{aligned}$$

which implies that

$$\lim_{n\to\infty} I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda_n)\in(a,b]\}\}I\{\lambda\le\Lambda_n<\lambda+\triangle\lambda\}$$
$$=I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\}\}I\{\lambda\le\Lambda<\lambda+\triangle\lambda\},\ \text{a.s.}$$

Then, by dominated convergence theorem,

$$\lim_{n\to\infty} E[I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda_n)\in(a,b]\}\}I\{\lambda\le\Lambda_n<\lambda+\triangle\lambda\}]$$
$$=E[I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\}\}I\{\lambda\le\Lambda<\lambda+\triangle\lambda\}],$$

or, equivalently,

$$\lim_{n\to\infty} P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda_n)\in(a,b]\},\lambda\le\Lambda_n<\lambda+\triangle\lambda)$$
$$=P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\},\lambda\le\Lambda<\lambda+\triangle\lambda). \tag{25}$$

Since Λ_n only depends on Λ and Λ is independent of the Poisson process $\{M(t)\}$, Λ_n is also independent of $\{M(t)\}$ and $\hat{c}_{k,T}(r)$, for any constant $r>0$. Therefore,

$$P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda_n)\in(a,b]\},\lambda\le\Lambda_n<\lambda+\triangle\lambda)$$
$$=\sum_{\lambda\le r<\lambda+\triangle\lambda} P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda_n)\in(a,b]\},\Lambda_n=r)$$
$$=\sum_{\lambda\le r<\lambda+\triangle\lambda} P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(r)\in(a,b]\},\Lambda_n=r)$$
$$=\sum_{\lambda\le r<\lambda+\triangle\lambda} P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(r)\in(a,b]\})P(\Lambda_n=r)$$
$$=\sum_{\lambda\le r<\lambda+\triangle\lambda} s_{L,U}(r)P(\Lambda_n=r)$$
$$=E[s_{L,U}(\Lambda_n)I\{\lambda\le\Lambda_n<\lambda+\triangle\lambda\}]. \tag{26}$$

It follows from (24) and (21) that

$$\lim_{n\to\infty} s_{L,U}(\Lambda_n)I\{\lambda\le\Lambda_n<\lambda+\triangle\lambda\}$$
$$=s_{L,U}(\Lambda)I\{\lambda\le\Lambda<\lambda+\triangle\lambda\},\ \text{a.s.,}$$

and then by dominated convergence theorem, since $|s_{L,U}(\Lambda_n)|\le 1$,

$$\lim_{n\to\infty} E[s_{L,U}(\Lambda_n)I\{\lambda\le\Lambda_n<\lambda+\triangle\lambda\}]$$
$$=E[s_{L,U}(\Lambda)I\{\lambda\le\Lambda<\lambda+\triangle\lambda\}]. \tag{27}$$

Thus, by (25), (26), and (27),

$$P(\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\},\lambda\le\Lambda<\lambda+\triangle\lambda)$$
$$=E[s_{L,U}(\Lambda)I\{\lambda\le\Lambda<\lambda+\triangle\lambda\}].$$

Since the Borel algebra $\mathcal{B}(\mathbb{R}^{++})$ can be generated by $\{[c,d):0<c<d\}$, it follows that

$$E[I\{\bigcap_{T=L}^{U}\{\hat{c}_{k,T}(\Lambda)\in(a,b]\}\}I\{\Lambda\in A\}]=E[s_{L,U}(\Lambda)I\{\Lambda\in A\}],$$
$$\forall A\in\mathcal{B}(\mathbb{R}^{++}),$$

so according to the definition of conditional expectation, (23) holds.

For constant $\lambda>0$ and integers $L<U$, define

$$h_{L,U,n}(\lambda):=P(\bigcap_{T=L}^{U}\{|\hat{c}_{k,T}(\lambda)|<\frac{1}{n}\}),$$
$$h_{L,n}(\lambda):=P(\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\lambda)|<\frac{1}{n}\}).$$

Since $\bigcap_{T=L}^{U}\{|\hat{c}_{k,T}(\lambda)|<\frac{1}{n}\}$ is a decreasing sequence of events when $U\to\infty$, it follows that

$$\lim_{U\to\infty} h_{L,U,n}(\lambda)=h_{L,n}(\lambda)\ \text{for any}\ \lambda>0,$$

and therefore,

$$\lim_{U\to\infty} h_{L,U,n}(\Lambda)=h_{L,n}(\Lambda),\ \text{a.s.} \tag{28}$$

Since $|h_{L,U,n}(\Lambda)|\le 1$, it follows from (23), (28), and the dominated convergence theorem that

$$P(\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})=\lim_{U\to\infty}P(\bigcap_{T=L}^{U}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})$$
$$=\lim_{U\to\infty}E[P(\bigcap_{T=L}^{U}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\}\mid\Lambda)]=\lim_{U\to\infty}E[h_{L,U,n}(\Lambda)]\ (\text{by (23)})$$
$$=E[h_{L,n}(\Lambda)]\ (\text{by (28) and dominated convergence theorem}). \tag{29}$$

By Lemma 3.1, for any $\lambda>0$, $\lim_{T\to\infty}\hat{c}_{k,T}(\lambda)=0$, a.s. Hence, for any $\lambda>0$, and any $n>0$,

$$1=P(\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\lambda)|<\frac{1}{n}\})=\lim_{L\to\infty}P(\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\lambda)|<\frac{1}{n}\})$$
$$=\lim_{L\to\infty}h_{L,n}(\lambda).$$

Therefore, for any $n>0$,

$$\lim_{L\to\infty}h_{L,n}(\Lambda)=1,\ \text{a.s.}$$

Since $|h_{L,n}(\Lambda)|\le 1$, it follows from the dominated convergence theorem that

$$\lim_{L\to\infty}E[h_{L,n}(\Lambda)]=1. \tag{30}$$

It follows from (29) and (30) that

$$\lim_{L\to\infty}P(\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})=1,\ \text{for}\ \forall n>0,$$

or, equivalently,

$$P(\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})=1,\ \text{for}\ \forall n>0. \tag{31}$$

Finally, by (31),

$$P(\lim_{T\to\infty}\hat{c}_{k,T}(\Lambda)=0)=P(\bigcap_{n=1}^{\infty}\{\exists L,\forall T\ge L,|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})$$
$$=P(\bigcap_{n=1}^{\infty}\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})=\lim_{n\to\infty}P(\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}(\Lambda)|<\frac{1}{n}\})$$
$$=1,$$

which completes the proof for (14).

(i.ii) Second, we prove (15) and (16). For any $\lambda > 0$, let $m_t(\lambda)$ be defined in (9). Define

$$g_{L,U,n}(\lambda) := P(\bigcap_{T=L}^{U}\{|\frac{1}{T}\sum_{t=1}^{T} m_t(\lambda)^2 - \delta\lambda - \delta^2\lambda^2| < \frac{1}{n}\}), \quad (32)$$

$$g_{L,n}(\lambda) := P(\bigcap_{T=L}^{\infty}\{|\frac{1}{T}\sum_{t=1}^{T} m_t(\lambda)^2 - \delta\lambda - \delta^2\lambda^2| < \frac{1}{n}\}). \quad (33)$$

Then, since $\bigcap_{T=L}^{U}\{|\frac{1}{T}\sum_{t=1}^{T} m_t(\lambda)^2 - \delta\lambda - \delta^2\lambda^2| < \frac{1}{n}\}$ is a decreasing sequence of events as $U \to \infty$, it follows that

$$\lim_{U\to\infty} g_{L,U,n}(\lambda) = g_{L,n}(\lambda), \ \forall\lambda > 0,$$

and hence

$$\lim_{U\to\infty} g_{L,U,n}(\Lambda) = g_{L,n}(\Lambda), \ \text{a.s.},$$

which, in combination with dominated convergence theorem and the fact that $|g_{L,U,n}(\Lambda)| \le 1$, implies that

$$\lim_{U\to\infty} E[g_{L,U,n}(\Lambda)] = E[g_{L,n}(\Lambda)]. \quad (34)$$

By similar argument for proving (23), we can show that

$$P(\bigcap_{T=L}^{U}\{|\frac{1}{T}\sum_{t=1}^{T} m_t^2 - \delta\Lambda - \delta^2\Lambda^2| < \frac{1}{n}\} \mid \Lambda) = g_{L,U,n}(\Lambda). \quad (35)$$

Since $m_t(\lambda), t = 1, 2, \ldots$, are i.i.d., it follows from law of large numbers that

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} m_t(\lambda)^2 = E[m_t(\lambda)^2] = \delta\lambda + \delta^2\lambda^2, \ \text{a.s.} \quad (36)$$

Hence, for any $n > 0$,

$$\begin{aligned}1 &= P(\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\frac{1}{T}\sum_{t=1}^{T} m_t(\lambda)^2 - \delta\lambda - \delta^2\lambda^2| < \frac{1}{n}\})\\ &= \lim_{L\to\infty} P(\bigcap_{T=L}^{\infty}\{|\frac{1}{T}\sum_{t=1}^{T} m_t(\lambda)^2 - \delta\lambda - \delta^2\lambda^2| < \frac{1}{n}\})\\ &= \lim_{L\to\infty} g_{L,n}(\lambda).\end{aligned}$$

Therefore, for any $n > 0$,

$$\lim_{L\to\infty} g_{L,n}(\Lambda) = 1, \ \text{a.s.}$$

Since $|g_{L,n}(\Lambda)| \le 1$, it follows from the dominated convergence theorem that

$$\lim_{L\to\infty} E[g_{L,n}(\Lambda)] = 1. \quad (37)$$

Finally,

$$\begin{aligned}&P(\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} m_t^2 = \delta\Lambda + \delta^2\Lambda^2)\\ =&P(\bigcap_{n=1}^{\infty}\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\frac{1}{T}\sum_{t=1}^{T} m_t^2 - \delta\Lambda - \delta^2\Lambda^2| < \frac{1}{n}\})\\ =& \lim_{n\to\infty} P(\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\frac{1}{T}\sum_{t=1}^{T} m_t^2 - \delta\Lambda - \delta^2\Lambda^2| < \frac{1}{n}\})\\ =& \lim_{n\to\infty}\lim_{L\to\infty} P(\bigcap_{T=L}^{\infty}\{|\frac{1}{T}\sum_{t=1}^{T} m_t^2 - \delta\Lambda - \delta^2\Lambda^2| < \frac{1}{n}\})\\ =& \lim_{n\to\infty}\lim_{L\to\infty}\lim_{U\to\infty} P(\bigcap_{T=L}^{U}\{|\frac{1}{T}\sum_{t=1}^{T} m_t^2 - \delta\Lambda - \delta^2\Lambda^2| < \frac{1}{n}\})\\ =& \lim_{n\to\infty}\lim_{L\to\infty}\lim_{U\to\infty} E[P(\bigcap_{T=L}^{U}\{|\frac{1}{T}\sum_{t=1}^{T} m_t^2 - \delta\Lambda - \delta^2\Lambda^2| < \frac{1}{n}\} \mid \Lambda)]\\ =& \lim_{n\to\infty}\lim_{L\to\infty}\lim_{U\to\infty} E[g_{L,U,n}(\Lambda)] \ (\text{by (35)})\\ =& \lim_{n\to\infty}\lim_{L\to\infty} E[g_{L,n}(\Lambda)] \ (\text{by (34)})\\ =&1 \ (\text{by (37)}),\end{aligned} \quad (38)$$

which completes the proof of (15).

Noting that

$$\frac{1}{T}\sum_{t=1}^{T}(m_t - \bar{m}_T)^2 = \frac{1}{T}\sum_{t=1}^{T} m_t^2 - (\bar{m}_T)^2.$$

By Proposition 4.2 in [8, p. 66], $\lim_{T\to\infty}\bar{m}_T = \delta\Lambda$, a.s., which in combination with (15) implies that (16) holds.

(i.iii) Third, we prove (17). It follows from (14) and (16) that

$$\lim_{T\to\infty} \hat{\rho}_{k,T} = \frac{\lim_{T\to\infty} \hat{c}_{k,T}}{\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}(m_t - \bar{m}_T)^2} = 0, \ \text{a.s.},$$

which completes the proof.

(ii) Let $N(t)$ be any mixed Poisson process that may not have the representation of (2). We have

$$\begin{aligned}&P(\lim_{T\to\infty} \hat{c}_{k,T} = 0) = P(\bigcap_{n=1}^{\infty}\{\exists L, \forall T \ge L, |\hat{c}_{k,T}| < \frac{1}{n}\})\\ =&P(\bigcap_{n=1}^{\infty}\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}| < \frac{1}{n}\}) = \lim_{n\to\infty} P(\bigcup_{L=1}^{\infty}\bigcap_{T=L}^{\infty}\{|\hat{c}_{k,T}| < \frac{1}{n}\})\\ =& \lim_{n\to\infty}\lim_{L\to\infty}\lim_{U\to\infty} P(\bigcap_{T=L}^{U}\{|\hat{c}_{k,T}| < \frac{1}{n}\}).\end{aligned}$$

Therefore, $P(\lim_{T\to\infty}\hat{c}_{k,T} = 0)$ only depends on the finite dimensional distributions of $\{m_t\}$ but not on the specific representation of $N(t)$. Since (14) holds for the specific mixed Poisson process defined in (2), (14) also holds for any mixed Poisson process $N(t)$. Similarly, $P(\lim_{T\to\infty}\hat{\rho}_{k,T} = 0)$ only depends on the finite dimensional distributions of $\{m_t\}$ but not on the specific representation of $N(t)$. Hence, (17) also holds for any mixed Poisson process $N(t)$ because it holds for the specific mixed Poisson process defined in (2). This completes the proof. □

4. Conclusion

Compared with a homogeneous Poisson process, a mixed Poisson process has a distinctive feature that it has strictly stationary but positively correlated increments. Although it holds under some conditions that the sample autocovariance and autocorrelation of a weakly stationary time series converge to its population counterparts, the increments of a mixed Poisson process are one of the interesting exceptions. More precisely, we show that, surprisingly, the sample autocovariance and autocorrelation at any lag of the increments of a mixed Poisson process converge to zero almost surely as the sample size goes to infinity. As a result, the sample autocovariance or autocorrelation cannot be used in the method of moments for parameter estimation of mixed Poisson processes.

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