Nonparametric Quantile Regressions for Panel Data Models with Large T

Liang Chen
Peking University

Abstract

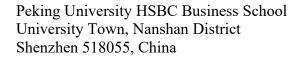This paper considers panel data models where the conditional quantiles of the dependent variables are additively separable as unknown functions of the regressors and the individual effects. We propose two estimators of the quantile partial effects while controlling for the individual heterogeneity. The first estimator is based on local linear quantile regressions, and the second is based on local linear smoothed quantile regressions, both of which are easy to compute in practice. Within the large T framework, we provide sufficient conditions under which the two estimators are shown to be asymptotically normally distributed. In particular, for the first estimator, it is shown that $N \ll T^{\frac{2}{d+4}}$ is needed to ignore the incidental-parameter biases, where d is the dimension of the regressors. For the second estimator, we are able to derive the analytical expression of the asymptotic biases under the assumption that $N \asymp Th^d$, where h is the bandwidth parameter in local linear approximations. Our theoretical results provide the basis of using split-panel jackknife for bias corrections. A Monte Carlo simulation shows that the proposed estimators and the bias-correction method perform well in finite samples.

Keywords: Panel data, quantile regressions, incidental parameters, jackknife, bias correction.
JEL Classification: C14, C31, C33

Peking University HSBC Business School
University Town, Nanshan District
Shenzhen 518055, China

PHBS
北京大学汇丰商学院

Nonparametric Quantile Regressions for Panel Data Models with Large T

Liang Chen[*]

Peking University HSBC Business School

January 5, 2021

Abstract

This paper considers panel data models where the conditional quantiles of the dependent variables are additively separable as unknown functions of the regressors and the individual effects. We propose two estimators of the quantile partial effects while controlling for the individual heterogeneity. The first estimator is based on local linear quantile regressions, and the second is based on local linear smoothed quantile regressions, both of which are easy to compute in practice. Within the large T framework, we provide sufficient conditions under which the two estimators are shown to be asymptotically normally distributed. In particular, for the first estimator, it is shown that $N \ll T^{\frac{2}{d+4}}$ is needed to ignore the incidental-parameter biases, where d is the dimension of the regressors. For the second estimator, we are able to derive the analytical expression of the asymptotic biases under the assumption that $N \asymp Th^d$, where h is the bandwidth parameter in local linear approximations. Our theoretical results provide the basis of using split-panel jackknife for bias corrections. A Monte Carlo simulation shows that the proposed estimators and the bias-correction method perform well in finite samples.

Keywords: Panel data, quantile regressions, incidental parameters, jackknife, bias correction.
JEL codes: C14, C31, C33.

1 Introduction

This paper studies the estimation of nonparametric quantile panel data models. To facilitate the discussion, consider the following model:

$$Y_{it} = Q(X_{it}, \alpha_i, \epsilon_{it}), \text{ for } i = 1, \ldots, N; t = 1, \ldots, T, \tag{1}$$

where $Y_{it} \in \mathbb{R}$ is the observed dependent variable, $X_{it} \in \mathcal{X} \subset \mathbb{R}^d$ is the observed regressors, $\alpha_i \in \mathbb{R}$ is the unobserved individual effect representing individual heterogeneity, and $\epsilon_{it}|(X_{it}, \alpha_i) \sim \mathcal{U}(0,1)$. Similar models have also been studied by Altonji and Matzkin (2005) and Chernozhukov et al. (2013) under different assumptions. Assuming that the mapping $\tau \mapsto Q(x, a, \tau)$ is strictly increasing for almost all (x, a) in the support of (X_{it}, α_i), then almost surely,

$$\mathsf{Q}_{Y_{it}}[\tau | X_{it} = x, \alpha_i = a] = Q(x, a, \tau) \stackrel{\text{def}}{=} Q_\tau(x, a),$$

where $\mathsf{Q}_{Y_{it}}[\tau|\cdot]$ denotes the τ-th conditional quantile of Y_{it}. Our main object of interest is the *quantile partial effects* (QPE, hereafter) of X_{it} on Y_{it} while controlling for the individual effects, i.e., $\partial Q_\tau(x, a)/\partial x$ for $\tau \in (0, 1)$.

Recent development in the literature of quantile panel data models with large T, including Koenker (2004), Lamarche (2010), Galvao and Montes-Rojas (2010), Galvao (2011), Canay (2011), Kato et al. (2012) and Galvao and Kato (2016), has mainly focused on the linear models where $Q_\tau(x, a) = \beta(\tau)'x + \lambda_\tau(a)$. This linearity specification for $Q_\tau(x, a)$ is convenient for constructing estimators of the QPE based on quantile regressions and analyzing their asymptotic properties, but it entails two possibly strong restrictions. First, in these models, $\partial Q_\tau(x, a)/\partial x = \beta(\tau)$, i.e., the QPE is homogeneous across x and a. Second, the linearity assumption on $Q_\tau(x, a)$ usually impose strong restrictions on the regressors. For example, consider location-scale shifting models: $Y_{it} = \beta'X_{it} + \alpha_i + g(X_{it}) \cdot \epsilon_{it}$, where ϵ_{it} is independent of (X_{it}, α_i). In order to have $Q_\tau(x, a)$ linear in x for all τ, we need $g(x) = \gamma'x > 0$ for some $\gamma \in \mathbb{R}^d$ and almost all x in the support of X_{it}. Thus, for $d = 1$, X_{it} must be positive almost surely if $\gamma > 0$.

To overcome the limitations of the linearity assumption, in this paper, we consider the following more general specification:

$$Q_\tau(x, a) = q_\tau(x) + \lambda_\tau(a), \tag{2}$$

which is a separable *nonparametric* model in the sense that q_τ and λ_τ are both unknown functions. In this case,

$$\partial Q_\tau(x, a)/\partial x = \partial q_\tau(x)/\partial x \stackrel{\text{def}}{=} \beta_\tau(x).$$

Thus, the QPE is allowed to be heterogeneous across x. Two estimators of $\beta_\tau(x)$ are proposed.

The first one is based on *local linear quantile regressions* (LLQR, hereafter) and the second one is based on *local linear smoothed quantile regressions* (LLSQR, hereafter). The main advantage of the proposed estimators is that computationally, they are as efficient as the estimators of Kato et al. (2012) and Galvao and Kato (2016) for linear quantile panel models.

Despite being computationally simple, analyzing the asymptotic properties of the LLQR estimator and the LLSQR estimator in the large T framework is a nontrivial task, mainly due to the well-known problem of "incidental parameters" — see and Lancaster (2000), Hahn and Newey (2004) and Fernández-Val and Weidner (2018). Another major contribution of this paper is that it provides a set of regularity conditions under which the proposed estimators are shown to be asymptotically normally distributed. In particular, for the LLQR estimator, the incidental-parameter biases are hard to characterize (see the discussions of Kato et al. 2012) and we need $N \ll T^{\frac{2}{d+4}}$ to ignore the asymptotic biases. On the other hand, under the assumption that $N \asymp Th^d$ (h is the bandwidth parameter in the local linear regression), we are able to derive the asymptotic bias of the LLSQR estimator for the boundary points of $\mathcal{X}$. Interestingly, the LLSQR estimator for the interior points of $\mathcal{X}$ are shown to be free of asymptotic bias. Moreover, our asymptotic analysis provides the theoretical basis of using split-panel jackknife (see Dhaene and Jochmans 2015) for bias corrections.

Other Related Literature

As pointed out by Arellano and Bonhomme (2011), the identification of nonlinear panel data models with fixed T is a nontrivial problem. Similarly, in the "small T" framework, the identification of $\partial Q_\tau(x, a)/\partial x$ is not straightforward. Invoking the result of Hu and Schennach (2008), one can show that for $T = 3$, if $\epsilon_{i1}, \epsilon_{i2}$ and ϵ_{i3} are mutually independent conditional on $X_i \stackrel{\text{def}}{=} (X_{i1}, \ldots, X_{iT})'$ and some other high level conditions are satisfied, the general model (1) is nonparametrically identified, i.e., all the conditional densities $\mathsf{f}_{Y_{it}|X_i, \alpha_i}$ for $t = 1, 2, 3$ and $\mathsf{f}_{\alpha_i|X_i}$ are identified (see Proposition 2.1 of Arellano and Bonhomme 2016). Given this result, the identification of $\partial Q_\tau(x, a)/\partial x$ follows easily. Evdokimov (2010) considers a separable model where $Q(X_{it}, \alpha_i, \epsilon_{it}) = m(X_{it}, \alpha_i) + U_{it}$ and $U_{it} \stackrel{\text{def}}{=} U(X_{it}, \epsilon_{it})$. In this model, $Q_\tau(x, a) = m(x, a) + \mathsf{Q}_{U_{it}}[\tau|X_{it} = x]$. For $T = 2$, Evdokimov (2010) provides sufficient conditions for the identification of $m(x, a)$ and $\mathsf{f}_{U_{it}|X_{it}}$, which implies the identification of $\partial Q_\tau(x, a)/\partial x$. Yan and Li (2018) considers a similar model with $Q(X_{it}, \alpha_i, \epsilon_{it}) = m(X_{it}) + \alpha_i + \sigma(X_{it})\epsilon_{it}$. They propose a multiple-step estimator of the conditional quantile function: $m(x) + \sigma(x)\mathsf{Q}_\epsilon(\tau)$, but no asymptotic theory was provided for this estimator. Moreover, varying-coefficients quantile panel models where $Q_\tau(x, a) = \beta_\tau(x_2)'x_1 + a$ and $x = (x_1', x_2')'$ is studied by Su and Hoshino (2016) and Cai et al. (2018).

The identification of the *quantile treatment effects* (QTE) in nonseparable panels with fixed T is considered by Chernozhukov et al. (2013) and Chernozhukov et al. (2015). Note that the QTE considered in these papers is the derivative of the *quantile structural function*: $Q_\tau^*(x)$,

which is defined by $P[Y_{it} \leq Q^*_\tau(x)|X_{it} = x] = \tau$. Therefore, it is obvious that $Q^*_\tau(x) \neq Q_\tau(x, a)$, and the QTE is different from the QPE. More recently, Graham et al. (2018) considers the case where $Q^*_\tau(x) = \beta_\tau(x)'x$ and focuses on the identification and estimation of the average conditional quantile effects (ACQEs) defined as $\mathbb{E}[\beta_\tau(X_{it})]$.

Last but not least, this paper extends a large literature on nonparametric quantile regressions (see Chaudhuri et al. 1991, Fan et al. 1994, Yu and Jones 1998, Honda 2000, Su and Ullah 2009, Qu and Yoon 2015, etc.) to panel data models with fixed effects.

Structure of the Paper

The rest of the paper is organized as follows: Section 2 introduces the models and provides some illustrative examples. Section 3 defines the estimators, whose asymptotic properties are established in Section 4. In Section 5, A Monte Carlo simulation is used to evaluate the performance of the proposed estimators and the bias-correction method. Finally, Section 6 concludes. All the proofs are collected in the appendix.

2 The Model and Some Examples

Specification (2) implies the following panel data model:

$$Y_{it} = q_\tau(X_{it}) + \lambda_\tau(\alpha_i) + u_{it}(\tau), \tag{3}$$

where the error terms satisfy the following quantile restrictions:

$$P[u_{it}(\tau) \leq 0|X_{it}, \alpha_i] = \tau.$$

It follows that the conditional quantile of the outcomes Y_{it} given the observed covariates X_{it} and the individual effect α_i can be written as

$$\mathsf{Q}_{Y_{it}}[\tau|X_{it} = x, \alpha_i = a] = Q_\tau(x, a) = q_\tau(x) + \lambda_\tau(a),$$

and as discussed in the introduction, our main object of interest is the QPE: $\beta_\tau(x) = \dot{q}_\tau(x)$[1] for all $x \in \mathcal{X} \subset \mathbb{R}^d$ and all $\tau \in \mathcal{T}$, where $\mathcal{T}$ is a compact subset of $[0, 1]$.

---

[1]To simply the notations we use $\dot{q}_\tau(x)$ and $\ddot{q}_\tau(x)$ to denote the first and second order derivatives of $q_\tau(\cdot)$ respectively.

Consider the following 3 examples:

$$\text{Example 1:} \qquad Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + \gamma X_{it}^2} \cdot \epsilon_{it}$$

$$\text{Example 2:} \qquad Y_{it} = \beta X_{it} + \alpha_i + \left(\sqrt{1 + \gamma X_{it}^2} + \sqrt{1 + \theta \alpha_i^2} \right) \cdot \epsilon_{it}$$

$$\text{Example 3:} \qquad Y_{it} = \beta X_{it} + \alpha_i + X_{it}\alpha_i + \sqrt{1 + \gamma X_{it}^2} \cdot \epsilon_{it}$$

where ϵ_{it} is independent of X_{it}, α_i with quantile function Q_ϵ. It follows that

$$\text{Example 1:} \qquad Q_\tau(x, a) = \underbrace{\beta x + \sqrt{1 + \gamma x^2} \cdot \mathsf{Q}_\epsilon(\tau)}_{q_\tau(x)} + \underbrace{a}_{\lambda_\tau(a)}.$$

$$\text{Example 2:} \qquad Q_\tau(x, a) = \underbrace{\beta x + \sqrt{1 + \gamma x^2} \cdot \mathsf{Q}_\epsilon(\tau)}_{q_\tau(x)} + \underbrace{a + \left(\sqrt{1 + \theta a^2} \right) \cdot \mathsf{Q}_\epsilon(\tau)}_{\lambda_\tau(a)}.$$

$$\text{Example 3:} \qquad Q_\tau(x, a) = \beta x + a + xa + \sqrt{1 + \gamma x^2} \cdot \mathsf{Q}_\epsilon(\tau).$$

Note that both Example 1 and Example 2 are nested by our model. In particular, the function $\lambda_\tau(\cdot)$ in Example 1 are invariant across $\tau \in (0, 1)$. Example 3 is not nested by our model, since the conditional quantile function is not additive separable as functions of x and a. Our model implies that the QPE is a function of X_{it} only, while in Example 3 the QPE depends on both X_{it} and α_i.

3 The Estimators

Suppose that we have a random sample of (Y_{it}, X_{it}) for $i = 1, \ldots, N$ and $t = 1, \ldots, T$, where the realized values of the individual effects are $(\alpha_{01}, \ldots, \alpha_{0N})$. We follow a fixed effects approach, treating $(\lambda_{01,\tau}, \ldots, \lambda_{0N,\tau}) \stackrel{\text{def}}{=} (\lambda_\tau(\alpha_{01}), \ldots, \lambda_\tau(\alpha_{0N}))$ as fixed parameters, and consider the asymptotic framework where both dimensions of the panel data go to infinity, i.e., $N, T \to \infty$.

Focus on a single point $x \in \mathcal{X}$. Expanding $q_\tau(X_{it})$ around x, we have

$$q_\tau(X_{it}) = q_\tau(x) + \dot{q}_\tau(x)'(X_{it} - x) + 0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R_\tau(x, X_{it})$$

where $R_\tau(x, X_{it})$ is the remainder term. It follows that

$$Y_{it} = \lambda_{0i,\tau} + q_\tau(x) + \dot{q}_\tau(x)'(X_{it} - x) + 0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R_\tau(x, X_{it}) + u_{it}(\tau),$$

The above representation motivates that following LLQR estimator for $\eta_{0i,\tau}(x) = \lambda_{0i,\tau} + q_\tau(x)$

and $\beta_\tau(x)$:

$$(\hat{\eta}_{1,\tau}(x), \ldots, \hat{\eta}_{N,\tau}(x), \hat{\beta}_\tau(x)) = \underset{\eta_1, \ldots, \eta_N, \beta}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho_\tau(Y_{it} - \eta_i - (X_{it} - x)'\beta) \cdot K\left(\frac{X_{it} - x}{h}\right)$$

where $\rho_\tau(u) = (\tau - \mathbf{1}(u \leq 0))u$ is the check function, $K(\cdot)$ is a multivariate kernel function, and h is a bandwidth parameter. Note that

$$
\begin{aligned}
&\rho_\tau(Y_{it} - \eta_i - (X_{it} - x)'\beta) \cdot K((X_{it} - x)/h) \\
={}& [\tau - \mathbf{1}(Y_{it} \leq \eta_i + (X_{it} - x)'\beta)] \cdot (Y_{it} - \eta_i - (X_{it} - x)'\beta) \cdot K((X_{it} - x)/h) \\
={}& [\tau - \mathbf{1}(\tilde{Y}_{it} \leq \eta_i K_{it} + \tilde{X}'_{it}\beta)] \cdot (\tilde{Y}_{it} - \eta_i K_{it} - \tilde{X}'_{it}\beta) \\
={}& \rho_\tau(\tilde{Y}_{it} - \eta_i K_{it} - \tilde{X}'_{it}\beta),
\end{aligned}
$$

where $K_{it} = K((X_{it} - x)/h)$, $\tilde{Y}_{it} = Y_{it}K_{it}$, and $\tilde{X}_{it} = (X_{it} - x)K_{it}$. Thus, the LLQR estimator can be easily calculated by running a standard quantile regression of $\tilde{Y}_{it}$ on $\tilde{X}_{it}$ and N additional regressors: $\mathbf{1}(i = 1)K_{it}, \ldots, \mathbf{1}(i = N)K_{it}$, therefore it is very computationally efficient.

Inspired by Galvao and Kato (2016), we also consider the following LLSQR estimator:

$$
\begin{aligned}
&(\check{\eta}_{1,\tau}(x), \ldots, \check{\eta}_{N,\tau}(x), \check{\beta}_\tau(x)) \\
={}& \underset{\eta_1, \ldots, \eta_N, \beta}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\tau - G\left(\frac{Y_{it} - \eta_i - (X_{it} - x)'\beta}{b}\right)\right)(Y_{it} - \eta_i - (X_{it} - x)'\beta) \cdot K\left(\frac{X_{it} - x}{h}\right),
\end{aligned}
$$

where $G(z) = 1 - \int_{-\infty}^{z} g(u)du$, $g(\cdot)$ is a continuously differentiable function with support $[-1, 1]$, and b is a bandwidth parameter. The idea of smoothed quantile regression (see Amemiya 1982 and Horowitz 1998) is to approximate the non-smooth indicator function with a smooth cumulative distribution function.

4 Asymptotic Results

Before presenting the asymptotic results, it is useful to define some new notations. Let x_∂ be on the boundary of $\mathcal{X}$. The *boundary points* are defined as

$$x = x_\partial + ch \text{ for some } c \in \text{supp}(K),$$

and the domain for integration is defined as

$$\mathcal{B} = \{v \in \mathbb{R}^d : (c + v)h \in \mathcal{X}\} \cap \text{supp}(K).$$

Define

$$c_0 = \int_{\mathcal{B}} K(u)du, \ \mathcal{C}_1 = \int_{\mathcal{B}} uK(u)du, \ \mathcal{C}_2 = \int_{\mathcal{B}} uu'K(u)du, \ \mathcal{C} = \mathcal{C}_2 - \mathcal{C}_1\mathcal{C}_1'/c_0,$$

$$\bar{\mathcal{C}}_2 = \begin{pmatrix} c_0 & \mathcal{C}_1' \\ \mathcal{C}_1 & \mathcal{C}_2 \end{pmatrix}, \ d_0 = \int_{\mathcal{B}} K^2(u)du, \ \mathcal{D}_1 = \int_{\mathcal{B}} uK^2(u)du, \ \mathcal{K}_1 = \int uu'K(u)du, \ \mathcal{K}_2 = \int uu'K^2(u)du.$$

4.1 Asymptotic Distribution of the LLQR Estimator

Write u_{it} instead of $u_{it}(\tau)$ to simply the notations. Let B_ϵ be a neighbourhood of 0. We first impose the following assumptions:

(A1) $\mathcal{X}$ is compact.

(A2) (Y_{it}, X_{it}) are independent of (Y_{js}, X_{js}) for any $i \neq j$ or $t \neq s$. $(Y_{i1}, X_{i2}), \ldots, (Y_{iT}, X_{iT})$ are identically distributed for each i.

(A3) Let $f_{u,i}(\cdot|x)$ denote the conditional density of u_{it} given $X_{it} = x$ and let $f_{X,i}(\cdot)$ denote the density of X_{it}. There exists $c_2 > c_1 > 0$ such that $c_1 < f_{u,i}(0|x) < c_2$ and $c_1 < f_{X,i}(x) < c_2$ for all i and all $x \in \mathcal{X}$.

(A4) Define $f_{u,i}^{(1)}(c|x) = \partial f_{u,i}(c|x)/\partial c$ and $f_{X,i}^{(1)}(x) = \partial f_{X,i}(x)/\partial x$. There exists a $M > 0$ such that $|f_{u,i}^{(1)}(c|x)| < M$ for all $c \in B_\epsilon$ and $|f_{X,i}^{(1)}(x)|, |\partial f_{u,i}(c|x)/\partial x| < M$ for all $x \in \mathcal{X}$. Moreover, $|\partial^2 q_\tau(x)/\partial x_j \partial x_p| < M$, $|\partial^3 q_\tau(x)/\partial x_j \partial x_p \partial x_h| < M$ for all $j, p, h \leq d$.

(A5) The kernel function K has bounded support and $\int uK(u)du = 0$, $\int u_j u_p u_h K(u)du = 0$ for all $j, p, h \leq d$.

(A6) $c_0 > 0$, $\mathcal{K}_1 > 0$, $\bar{\mathcal{C}}_2 > 0$ and $\mathcal{C} > 0$, and that

$$0 < \sigma(x) \overset{\text{def}}{=} \frac{\lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} f_{X,i}(x)}{\lim_{N \to \infty} [N^{-1} \sum_{i=1}^{N} f_{X,i}(x) f_{u,i}(0|x)]^2} < \infty$$

for all $x \in \mathcal{X}$.

(A7) Let $N \asymp T^{c_N}$ and $h \asymp T^{-c_h}$ for some $c_N, c_h > 0$. Then $c_N < \frac{2}{d+4}$ and $\frac{1}{d+4} < c_h < \frac{1-2c_N}{d}$.

Remark 1.1: The above assumptions, except (A2) and (A7), are standard in the literature of local linear quantile regressions and quantile regressions. Note that we only need the existence and smoothness of the conditional density of u_{it} given X_{it}, thus the estimator is robust to heavy tails and outliers in u_{it}.

Remark 1.2: The independence assumption (A2) is also adopted by Kato et al. (2012) and it excludes time-invariant regressors. This independence assumption can be relaxed to allow for β-mixing on the time dimension along the line of Galvao and Kato (2016) at the cost of much lengthier proofs. Thus, to keep the proofs tractable, (A2) is maintained throughout the paper.

7

Remark 1.3: Assumption (A7) ensures that $\log N \ll Th^{d+4}$, $N \ll Th^{d+2}$, $NTh^{d+6} \to 0$, and $N^2 \ll Th^d$. These conditions are needed to prove Theorem 1 below. For example, for $d = 2$ and $c_N = 1/4$, we can choose $c_h = 1/5$. Note that due to the nonparametric nature of our estimator, the condition $N^2 \ll Th^d$ imposed here is stronger than the condition $N^2 \ll T$ required by Kato et al. (2012), since the order of the incidental-parameter bias is approximately $(Th^d)^{-3/4}$, while in Kato et al. (2012) the bias is approximately of order $T^{-3/4}$. Such conditions are hard to justify in practice, this is why we also consider the LLSQR estimator, whose asymptotic distribution can be established under more realistic assumptions about the relative sizes of N and T.

The following theorem gives the asymptotic distribution of the LLQR estimator.

Theorem 1. *Suppose that Assumptions (A1) to (A7) hold, then:*

(i) For any interior point x of $\mathcal{X}$, we have

$$\sqrt{NTh^{d+2}}\big[\hat{\beta}_\tau(x) - \beta_\tau(x)\big] \xrightarrow{d} \mathcal{N}\Big(0, \tau(1-\tau)\sigma(x)\mathcal{K}_1^{-1}\mathcal{K}_2\mathcal{K}_1^{-1}\Big).$$

(ii) For any boundary point x of $\mathcal{X}$, i.e., $x = ch$ for some $c > 0$ in the support of $K(\cdot)$, we have

$$\sqrt{NTh^{d+2}}\big[\hat{\beta}_\tau(x) - \beta_\tau(x) - hB^{(1)}\big] \xrightarrow{d} \mathcal{N}\Big(0, \tau(1-\tau)\sigma(0)\Omega\Big),$$

where

$$\Omega = \mathcal{C}^{-1}\left[\int_{\mathcal{B}}(u - \mathcal{C}_1/c_0)(u - \mathcal{C}_1/c_0)'K^2(u)du\right]\mathcal{C}^{-1} \text{ and } B^{(1)} = 0.5\mathcal{C}^{-1}\int_{\mathcal{B}}u'\ddot{q}_\tau(0)u\,(u - \mathcal{C}_1/c_0)\,K(u)du.$$

Remark 1.4: The LLQR estimator suffers from two types of biases: a bias due to the estimation of incidental parameters, and another one due to local linear approximations. As discussed in Remark 1.3, the first bias can be ignored at the expense of a very strong condition: $N \ll T^{\frac{2}{d+4}}$. The term $hB^{(1)}$ is the leading bias term in the local linear approximations (see Fan et al. 1994 for example). This bias can be further reduced by using local polynomial regressions. Note that $B^{(1)} = 0$ for the interior points, thus the leading bias term for the estimators of the interior points is $O(h^2)$.

Remark 1.5: In general, it is straightforward to construct consistent estimators of the asymptotic variances, since $\mathcal{K}_1, \mathcal{K}_2$ and Ω only depend on the kernel function $K(\cdot)$, and $\sigma(x)$ can be consistently estimated using standard nonparametric methods. In particular, if the distribution of (u_{it}, X_{it}) are identical across i, i.e., $f_{u,i}(0|x) = f_u(0|x)$ and $f_{X,i}(x) = f_X(x)$ for all i, then $\sigma(x) = (f_X(x))^{-1} \cdot (f_u(0|x))^{-2}$.

4.2 Asymptotic Distribution of the LLSQR Estimator

We impose the following assumptions:

(B1) Assumptions (A1) to (A6) hold.

(B2) $f_{u,i}(c|x)$ is $m+2$ times continuously differentiable in c for all $x \in \mathcal{X}$. Let $f_{u,i}^{(j)}(c|x) = \partial^j f_{u,i}(c|x)/\partial c^j$, then there exists some M such that $|f_{u,i}^{(j)}(c|x)| < M$ for all $j \leq m+2$.

(B3) $g(v)$ is a symmetric function with support $[-1,1]$ and $\int g(v)dv = 1$. For some positive integer $m \geq 4$, $\int v^j g(v)dv = 0$ for $j = 1, \ldots, m-1$, and $\int v^m g(v)dv < \infty$.

(B4) $N/(Th^d) \to \kappa^2$ for some $\kappa > 0$. $h \asymp T^{-c_h}$ and $b \asymp T^{-c_b}$ for some $c_h, c_b > 0$ that satisfy

$$c_h \in \left(\frac{1}{d+3}, \frac{1}{d+1} \right) \cup \left(\frac{c_b}{2}, \frac{mc_b}{2} \right) \cup \left(\frac{1-mc_b}{d}, \frac{1-3c_b}{m} \right) \tag{4}$$

$$\frac{2}{m(2+d)} < c_b < \frac{2}{d+6}. \tag{5}$$

Remark 2.1: Assumptions (B2) and (B3) are also imposed in Galvao and Kato (2016). In particular, we need $g(\cdot)$ to be a fourth (or higher) order kernel function. Assumption (B4) is new. Condition (4) implies that $Th^{d+1} \to \infty$, $Th^{d+3} \to 0$, $Th^d b^3 \to \infty$, $Th^d b^m \to 0$ and $b^m \ll h^2 \ll b$. These conditions will be used in the proof of Theorem 2. Moreover, $m \geq 4$ and (5) ensure that c_h lies in a non-empty set. For example, for $m = 4$ and $d = 2$, one can choose $c_b = 1/6$ and $c_h \in (1/5, 1/4)$.

The following theorem gives the asymptotic distribution of the LLSQR estimator.

Theorem 2. *Suppose that Assumptions (B1) to (B4) hold, then:*

(i) For any interior point x of $\mathcal{X}$, we have

$$\sqrt{NTh^{d+2}}\big[\check{\beta}_\tau(x) - \beta_\tau(x)\big] \xrightarrow{d} \mathcal{N}\Big(0, \tau(1-\tau)\sigma(x)\mathcal{K}_1^{-1}\mathcal{K}_2\mathcal{K}_1^{-1}\Big).$$

(ii) For any boundary point x of $\mathcal{X}$, i.e., $x = ch$ for some $c > 0$ in the support of $K(\cdot)$, we have

$$\sqrt{NTh^{d+2}}\big[\check{\beta}_\tau(x) - \beta_\tau(x) - hB^{(1)}\big] \xrightarrow{d} \mathcal{N}\Big(\kappa B^{(2)}, \tau(1-\tau)\sigma(0)\Omega\Big),$$

where

$$B^{(2)} = -\frac{\tau - 1/2}{\lim_{N\to\infty} N^{-1} \sum_{i=1}^N f_{X,i}(x) f_{u,i}(0|x)} \cdot \mathcal{C}^{-1}(\mathcal{D}_1/c_0 - \mathcal{C}_1 d_0/c_0^2).$$

Remark 2.2: It can be seen that the asymptotic distributions of the LLSQR estimators and the LLQR estimators are very similar, with one noticeable difference: the LLSQR estimator for the boundary points suffers from an asymptotic bias: $\kappa B^{(2)}$, which is the consequence of estimating incidental parameters. In the proof of Theorem 2, it is found that the incidental-parameter bias

of the LLSQR estimator for the boundary points is of order $(Th^d)^{-1}$ rather than T^{-1} — this is why we need $N \asymp Th^d$ to derive the analytical expression of the asymptotic bias. Interestingly, the LLSQR estimators for the boundary points at $\tau = 0.5$ and the LLSQR estimators for the interior points at all τs are all free of asymptotic biases. These findings are further confirmed by a Monte Carlo simulation in Section 5.

Remark 2.3: Theorem 2 provides the theoretical basis for bias corrections using the split-panel jackknife method proposed by Dhaene and Jochmans (2015). In particular, divide the whole sample into two subsamples: (Y_{it}, X_{it}) for $i = 1, \ldots, N; t = 1, \ldots, T/2$, and (Y_{it}, X_{it}) for $i = 1, \ldots, N; t = T/2 + 1, \ldots, T$, and let $\check{\beta}_{\tau,1}(x), \check{\beta}_{\tau,2}(x)$ denote the LLSQR estimators using the two subsamples respectively[2]. The bias-corrected estimator is simply given by

$$\check{\beta}_\tau^{bc}(x) = 2\check{\beta}_\tau(x) - 0.5[\check{\beta}_{\tau,1}(x) + \check{\beta}_{\tau,2}(x)].$$

Under Assumptions (B1) to (B4) we can show that for the boundary points,

$$\sqrt{NTh^{d+2}}\big[\check{\beta}_\tau^{bc}(x) - \beta_\tau(x) - hB^{(1)}\big] \overset{d}{\to} \mathcal{N}\Big(0, \tau(1-\tau)\sigma(0)\Omega\Big).$$

Remark 2.4: Assumption (B4) requires that $N \asymp Th^d$, which is much less restrictive than Assumption (A6) which imposes $N \ll \sqrt{Th^d}$. As discussed in Remark 2.1, for $d = 2$, Assumption (B4) admits the choice: $c_h = 1/4.5$ and therefore $N \asymp T^{5/9}$. Thus, given the nonparametric nature of the problem, Assumption (B4) is still more stringent than the usual assumption $N \asymp T$ imposed for nonlinear fixed-effects estimators (see Hahn and Newey 2004 and Fernández-Val and Weidner 2018).

5 A Monte Carlo Simulation

In this section, we evaluate the performance of the proposed estimators in finite samples using the following data generating process (DGP):

$$Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it},$$

where $X_{it} \sim i.i.d\ \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i \sim i.i.d\ \mathcal{N}(0,1)$. It is easy to see that $\beta_\tau(x) = 1 + \mathsf{Q}_\epsilon(\tau) \cdot x/\sqrt{1 + x^2}$, where ϵ_{it} are i.i.d with quantile function Q_ϵ. We consider two different distributions of ϵ_{it}: (i) $\mathcal{N}(0,1)$, and (ii) t distribution with 3 degrees of freedom, and compare the biases and mean-square errors (MSEs) of four different estimators: the LLQR estimator $\hat{\beta}_\tau$, the LLSQR estimator $\check{\beta}_\tau$, and the bias-corrected versions of these two estimators, denoted as

---

[2]The bandwidth parameter h should be the same for $\check{\beta}_\tau(x)$, $\check{\beta}_{\tau,1}(x)$ and $\check{\beta}_{\tau,2}(x)$.

$\hat{\beta}_\tau^{bc}$ and $\check{\beta}_\tau^{bc}$ respectively.

To same space, we only report the results for $N = T = 100$, $\tau = 0.25, 0.5, 0.75$ and $x = -2, -1.6, \ldots, 1.6, 2$. For all estimators, we choose $h = 0.8$, and for the LLSQR estimators, we choose $b = 0.5$ and consider the following fourth-order kernel function:

$$k(u) = \frac{105}{64} \left(1 - 5u^2 + 7u^4 - 3u^6\right) 1(|u| \le 1).$$

We have also tried other bandwidth values and find that the results is more sensitive to the choice of h than the choice of b.

Table 1 reports the results for $\tau = 0.25$ and $\epsilon_{it} \sim \mathcal{N}(0, 1)$ while Table 2 reports the results for $\tau = 0.25$ and $\epsilon_{it} \sim T(3)$. The results for $\tau = 0.5$ and $\tau = 0.75$ are reported in Table 3 to Table 6.

For $\tau = 0.25$, four conclusions can be drawn from the results in Tables 1 and 2. (i) The performance of $\hat{\beta}_\tau$ and $\check{\beta}_\tau$, in terms of biases and MSEs, are very close. (ii) As predicted by our Theorem 2, the bias-correction method significantly reduces the biases of the LLSQR estimators, especially at the boundary points (e.g., $|x| = 2, 1.6$). Interestingly, the bias-correction method can also effectively reduce the biases of the LLQR estimators. (iii) The bias-corrected estimators for the boundary points have much lower MSEs due to the large decrease in biases. (iv) The performance of the estimators are robust to heavy tails of ϵ_{it}. Similar conclusions are supported by the results for $\tau = 0.75$. However, for $\tau = 0.5$, the bias correction at the boundary points is not very effective — this is predicted by Theorem 2, which shows that the LLSQR estimator for the boundary points is free of asymptotic biases at $\tau = 0.5$ (see Remark 2.2).

6 Conclusion

To the best of our knowledge, this is the first paper that considers nonparametric quantile regressions in the context of large T panels. Our model is additively separable as unknown functions of the regressors and the individual effects, and it allows the QPE to be heterogeneous across individuals. We propose two estimators of the QPE based on local linear approximations, and establish their asymptotic distributions under a set of regularity assumptions. Our theoretical results highlight the importance of incidental-parameter biases and justify the use of convenient jackknife method to correct the asymptotic biases. The good performance of the bias-correction method in finite samples is confirmed using a Monte Carlo simulation.

Like any other nonparametric estimators, the choice of bandwidth is crucial in practice. In this paper we have focused on the theoretical conditions that the bandwidth parameters have to satisfy, but how to choose those bandwidths in practice is an important question that is left for further investigation.

A   Proofs of The Main Results

A.1   Proof of Theorem 1

To simply the notations, we suppress the dependence of the parameters on τ and x. For example, we write $\eta_{0i}, \hat{\eta}_i, \hat{\beta}, \beta_0$ instead of $\eta_{0i,\tau}(x), \hat{\eta}_{i,\tau}(x), \hat{\beta}_\tau(x), \beta_\tau(x)$. Moreover, we first define the following notations: $\phi = h\beta, \hat{\phi} = h\hat{\beta}, \phi_0 = h\beta_\tau(x), \theta_i = (\eta_i, \phi')', \theta = (\eta_1, \ldots, \eta_N, \phi')', \theta_{0i} = (\eta_{0i}, \phi_0')', \theta_0 = (\eta_{01}, \ldots, \eta_{0N}, \phi_0')', W_{it} = (1, (X_{it} - x)'/h)', \hat{\theta}_i = (\hat{\eta}_i, \hat{\phi}')', \hat{\theta} = (\hat{\eta}_1, \ldots, \hat{\eta}_N, \hat{\phi}')', \psi_\tau(u) = \mathbf{1}\{u \leq 0\} - \tau,$

$$S_{T,i}(\theta_i) = \frac{1}{Th^d} \sum_{t=1}^T \rho_\tau(Y_{it} - \theta_i' W_{it}) K_{it}, \quad \bar{S}_i(\theta_i) = \mathbb{E}[\rho_\tau((\theta_{0i} - \theta_i)' W_{it} + u_{it}) K_{it}/h^d]$$

and $S_{NT}(\theta) = N^{-1} \sum_{i=1}^N S_{T,i}(\theta_i), \bar{S}_N(\theta) = N^{-1} \sum_{i=1}^N \bar{S}_i(\theta_i)$.

Lemma 1. *Under Assumptions A1 to A6, we have $\|\hat{\phi} - \phi_0\|_1 = o_P(1)$ and $\max_{i \leq N} |\hat{\eta}_i - \eta_{0i}| = o_P(1)$.*

Proof. For any $\delta > 0$, define $B_i(\delta) = \{\theta_i : |\eta_i - \eta_{0i}| + \|\phi - \phi_0\|_1 \leq \delta\}$. For any $\bar{\theta}_i \in B_i^C(\delta)$, define $\tilde{\theta}_i = r_i \bar{\theta}_i + (1 - r_i)\theta_0$, where $r_i = \delta/(|\bar{\eta}_i - \eta_{0i}| + \|\bar{\phi} - \phi_0\|_1) < 1$. Note that $\tilde{\theta}_i$ is on the boundary of $B_i(\delta)$. By the convexity of $S_{T,i}(\theta_i)$ we have

$$S_{T,i}(\tilde{\theta}_i) \leq r_i S_{T,i}(\bar{\theta}_i) + (1 - r_i)S_{T,i}(\theta_{0i}), \text{ or } S_{T,i}(\tilde{\theta}_i) - S_{T,i}(\theta_{0i}) \leq r_i(S_{T,i}(\bar{\theta}_i) - S_{T,i}(\theta_{0i})).$$

Next, by the definition of the estimator, we have $S_{T,i}(\hat{\theta}_i) \leq S_{T,i}(\theta_{0i})$ for some $i \leq N$. Thus, if $\|\hat{\phi} - \phi_0\|_1 > \delta$, then $\hat{\eta}_i \in B_i^C(\delta)$, which (by the above inequality) implies that

$$(S_{T,i}(\tilde{\theta}_i) - S_{T,i}(\theta_{0i}))/r_i \leq S_{T,i}(\hat{\theta}_i) - S_{T,i}(\theta_{0i}) \leq 0.$$

Adding the subtracting terms, the above inequality can be written as

$$\bar{S}_i(\tilde{\theta}_i) - \bar{S}_i(\theta_{0i}) \leq S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - [S_{T,i}(\tilde{\theta}_i) - \bar{S}_i(\tilde{\theta}_i)].$$

For interior points, using Taylor expansion, we have for some $C > 0$ that does not depend on i, and small enough δ,

$$\begin{aligned}
\bar{S}_i(\tilde{\theta}_i) - \bar{S}_i(\theta_{0i}) &= \mathbb{E}[\psi_\tau(u_{it})W_{it}K_{it}/h^d](\tilde{\theta}_i - \theta_{0i}) \\
&\quad + 0.5 f_{u,i}(0|x)f_{X,i}(x)[(\tilde{\eta}_i - \eta_{0i})^2 + (\tilde{\phi} - \phi_0)'\mathcal{K}_1(\tilde{\phi} - \phi_0)](1 + o(1)) > C\delta^2(1 + o(1)),
\end{aligned}$$

where we have used the fact that $\mathbb{E}[\psi_\tau(u_{it})W_{it}K_{it}/h^d] = 0$. Similarly, for the boundary points, we have

$$\bar{S}_i(\tilde{\theta}_i) - \bar{S}_i(\theta_{0i}) = 0.5 f_{u,i}(0|x) f_{X,i}(x)(\tilde{\theta}_i - \theta_{0i})' \bar{C}_2(\tilde{\theta}_i - \theta_{0i})(1 + o(1)) > C\delta^2(1 + o(1)).$$

Thus, if follows from the union bound that

$$P[\|\hat{\phi} - \phi_0\|_1 > \delta] \leq \sum_{i=1}^{N} P\big[S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - (S_{T,i}(\tilde{\theta}_i) - \bar{S}_i(\tilde{\theta}_i)) > C\delta^2(1 + o(1))\big].$$

To prove $\|\hat{\phi} - \phi_0\|_1 = o_P(1)$, it then suffices to show that for any $\epsilon > 0$

$$\max_{i \leq N} P\left[\sup_{\theta_i \in B_i(\delta)} |S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - (S_{T,i}(\theta_i) - \bar{S}_i(\theta_i))| > \epsilon\right] = o(N^{-1}). \qquad \text{(A.1)}$$

Note that

$$
\begin{aligned}
& S_{T,i}(\theta_i) - \bar{S}_i(\theta_i) \\
= {}& \frac{1}{Th^d} \sum_{t=1}^{T} \rho_\tau(Y_{it} - \theta_i'W_{it})K_{it} - \frac{1}{Th^d} \sum_{t=1}^{T} \mathbb{E}[\rho_\tau((\theta_{0i} - \theta_i)'W_{it} + u_{it})K_{it}] \\
= {}& \underbrace{\frac{1}{Th^d} \sum_{t=1}^{T} \big[\rho_\tau(Y_{it} - \theta_i'W_{it}) - \rho_\tau((\theta_{0i} - \theta_i)'W_{it} + u_{it})\big]K_{it}}_{A_{T,i}(\theta_i)} \\
& + \underbrace{\frac{1}{Th^d} \sum_{t=1}^{T} \big[\rho_\tau((\theta_{0i} - \theta_i)'W_{it} + u_{it})K_{it} - \mathbb{E}[\rho_\tau((\theta_{0i} - \theta_i)'W_{it} + u_{it})K_{it}]\big]}_{B_{T,i}(\theta_i)},
\end{aligned}
$$

so

$$\sup_{\theta_i \in B_i(\delta)} |S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - (S_{T,i}(\theta_i) - \bar{S}_i(\theta_i))| \leq \sup_{\theta_i \in B_i(\delta)} 2|A_{T,i}(\theta_i)| + \sup_{\theta_i \in B_i(\delta)} |B_{T,i}(\theta_i) - B_{T,i}(\theta_{0i})|.$$

Write $Y_{it} = \theta_{0i}'W_{it} + 0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x)^2 + R(x, X_{it}) + u_{it}$, so from $\rho_\tau(u_1) - \rho_\tau(u_2) \leq$

13

$2|u_1 - u_2|,$

$$\sup_{\theta_i \in B_i(\delta)} |A_{T,i}(\theta_i)|$$

$$\leq \quad \frac{C_1}{Th^d} \sum_{t=1}^{T} \left(\|X_{it} - x\|_1^2 + \|X_{it} - x\|_1^3 \right) K_{it}$$

$$= \quad \frac{C_1}{Th^d} \sum_{t=1}^{T} \left\{ \left(\|X_{it} - x\|_1^2 + \|X_{it} - x\|_1^3 \right) K_{it} - \mathbb{E}\left[\left(\|X_{it} - x\|_1^2 + \|X_{it} - x\|_1^3 \right) K_{it} \right] \right\} + \bar{O}(h^2).$$

Next, consider $\sup_{\theta_i \in B_i(\delta)} |B_{T,i}(\theta_i) - B_{T,i}(\theta_{0i})|$. By compactness of $B_i(\delta)$, there exist a finite L and $\theta_i^{(1)}, \ldots, \theta_i^{(L)} \in B_i(\delta)$ such that $\|\theta_i^{(j)} - \theta_i^{(k)}\|_1 \leq \omega$ for all $j, k \leq L$. For some $C_2, C_3 > 0$

$$\sup_{\theta_i \in B_i(\delta)} |B_{T,i}(\theta_i) - B_{T,i}(\theta_{0i})| \leq \max_{j \leq L} |B_{T,i}(\theta_i^{(j)}) - B_{T,i}(\theta_{0i})|$$

$$+ \omega C_2 \frac{1}{Th^d} \sum_{t=1}^{T} \left[\|W_{it}\|_1 K_{it} - \mathbb{E}[\|W_{it}\|_1 K_{it}] \right] + \omega C_3 \mathbb{E}[\|W_{it}\|_1 K_{it}/h^d].$$

Take $\omega = \epsilon/(6C_3 M)$ where $M = \sup_i \mathbb{E}[\|W_{it}\|_1 K_{it}/h^d] < \infty$, then for large enough N, T, we have

$$P\left[\sup_{\theta_i \in B_i(\delta)} |S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - (S_{T,i}(\theta_i) - \bar{S}_i(\theta_i))| > \epsilon \right] \leq$$

$$P\left[\frac{C_1}{Th^d} \sum_{t=1}^{T} \left\{ \left(\|X_{it} - x\|_1^2 + \|X_{it} - x\|_1^3 \right) K_{it} - \mathbb{E}\left[\left(\|X_{it} - x\|_1^2 + \|X_{it} - x\|_1^3 \right) K_{it} \right] \right\} > \epsilon/10 C_1 \right]$$

$$+ \sum_{j=1}^{L} P\left[|B_{T,i}(\theta_i^{(j)}) - B_{T,i}(\theta_{0i})| > \epsilon/5 \right] + P\left[\frac{1}{Th^d} \sum_{t=1}^{T} \left[\|W_{it}\|_1 K_{it} - \mathbb{E}[\|W_{it}\|_1 K_{it}] \right] > 6C_3 M/5C_2 \right].$$

$$(\text{A.2})$$

Consider the last term on the RHS of (A.2), since $h\|W_{it}\|_1 K_{it}$ is uniformly bounded, and $\mathbb{E}\left[\|W_{it}\|_1^2 K_{it}^2 \right] = \bar{O}(h^d)$, by Bernstein's inequality, we have

$$P\left[\frac{1}{Th^d} \sum_{t=1}^{T} \left[\|W_{it}\|_1 K_{it} - \mathbb{E}[\|W_{it}\|_1 K_{it}] \right] > C \right] \leq e^{-\frac{Th^{2d}}{C_4(h^d + h^{d-1})}}$$

for some $C_4 > 0$. We can establish the same bound for the other two terms on the RHS of (A.2). Thus, (A.1) holds since our assumptions imply that $\log N/(Th^{d+1}) \to 0$. Thus, it follows that $\|\hat{\phi} - \phi_0\|_1 = o_P(1)$.

Now consider $\hat{\eta}_i$. By definition of the estimators, we have $S_{T,i}(\hat{\eta}_i, \hat{\phi}) \leq S_{T,i}(\eta_{0i}, \hat{\phi})$ for all

14

$i \leq N$. Therefore, if $|\hat{\eta}_i - \eta_{0i}| > \delta$ for some $i \leq N$, then we have $\hat{\theta}_i \in B_i^C(\delta)$, and

$$(S_{T,i}(\tilde{\theta}_i) - S_{T,i}(\theta_{0i}))/r_i \leq S_{T,i}(\hat{\theta}_i) - S_{T,i}(\theta_{0i}) \leq S_{T,i}(\eta_{0i}, \hat{\phi}) - S_{T,i}(\theta_{0i}).$$

where r_i and $\tilde{\theta}$ is as define above. Adding the subtracting terms, the above inequality can be written as

$$\bar{S}_i(\tilde{\theta}_i) - \bar{S}_i(\theta_{0i}) \leq S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - [S_{T,i}(\tilde{\theta}_i) - \bar{S}_i(\tilde{\theta}_i)] + r_i(S_{T,i}(\eta_{0i}, \hat{\phi}) - S_{T,i}(\theta_{0i})).$$

Thus, for any $\delta > 0$, we have

$$P\left[\max_{i \leq N} |\hat{\eta}_i - \eta_{0i}| > \delta\right] \leq \sum_{i=1}^{N} P\left[\sup_{\theta_i \in B_i(\delta)} |S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - (S_{T,i}(\theta_i) - \bar{S}_i(\theta_i))| > C_6\delta^2\right]$$

$$+ P\left[\max_{1 \leq i \leq N} \left|S_{T,i}(\eta_{0i}, \hat{\phi}) - S_{T,i}(\theta_{0i})\right| > C_7\delta^2\right] \quad \text{(A.3)}$$

for some $C_6, C_7 > 0$. The first term on the RHS of (A.3) was shown to be $o(1)$ above. To prove $\max_{i \leq N} |\hat{\eta}_i - \eta_{0i}| = o_P(1)$ it suffices to show that for any $\epsilon > 0$,

$$P\left[\max_{i \leq N} |S_{T,i}(\eta_{0i}, \hat{\phi}) - S_{T,i}(\theta_{0i})| > \epsilon\right] = o(1). \quad \text{(A.4)}$$

Note that

$$|S_{T,i}(\eta_{0i}, \hat{\phi}) - S_{T,i}(\theta_{0i})| \lesssim \|\hat{\phi} - \phi_0\|_1 \cdot \frac{1}{Th^d} \sum_{t=1}^{T} \|(X_{it} - x)/h\|_1 K_{it}.$$

Moreover,

$$\max_{1 \leq i \leq N} \frac{1}{Th^d} \sum_{t=1}^{T} \|(X_{it}-x)/h\|_1 K_{it} \leq \max_{1 \leq i \leq N} \frac{1}{Th^d} \sum_{t=1}^{T} \{\|(X_{it} - x)/h\|_1 K_{it} - \mathbb{E}[\|(X_{it} - x)/h\|_1 K_{it}]\}$$

$$+ \max_{1 \leq i \leq N} \mathbb{E}[\|(X_{it} - x)/h\|_1 K_{it}/h^d]. \quad \text{(A.5)}$$

Similar to the proof above, we can show that the first term on the RHS of (A.5) is $o_P(1)$, and the second term is $\max_{1 \leq i \leq N} f_{X,i}(x) + o(1) < \infty$ by our assumptions. Then (A.4) follows by $\|\hat{\phi} - \phi_0\|_1 = o_P(1)$. This concludes the proof. $\qquad \square$

Define

$$\bar{S}_i^{\eta_i}(\theta_i) = \mathbb{E}\left[[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \tau]\frac{K_{it}}{h^d}\right], \quad \bar{S}_i^{\phi}(\theta_i) = \mathbb{E}\left[[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \tau]\frac{X_{it} - x}{h}\frac{K_{it}}{h^d}\right].$$

Denote $\bar{S}_i^{\eta_i \eta_i}(\theta_i) = \partial \bar{S}_i^{\eta_i}(\theta_i)/\partial \eta_i$, and $\bar{S}_i^{\phi\phi}(\theta_i), \bar{S}_i^{\phi\eta_i}(\theta_i)$ etc. are defined in a similar fashion. The arguments of these functions are dropped when they are evaluated at θ_{0i}. Let $f_i(x) = f_{u,i}(0|x)f_{X,i}(x)$, $\bar{f}_N(x) = N^{-1}\sum_{i=1}^N f_i(x)$, $\bar{f}(x) = \lim_{N\to\infty} \bar{f}_N(x)$.

Lemma 2. *For a boundary point x, i.e., $x = ch$ for some c in the support of $K(\cdot)$, we have*

$$\bar{S}_i^{\phi\phi} = \mathcal{C}_2 f_i(0) + \bar{O}(h), \quad \bar{S}_i^{\phi\eta_i} = \mathcal{C}_1 f_i(0) + \bar{O}(h) \quad \bar{S}_i^{\eta_i \eta_i} = c_0 f_i(0) + \bar{O}(h).$$

For a interior point x, we have

$$\bar{S}_i^{\phi\phi} = \mathcal{K}_1 f_i(x) + \bar{O}(h), \quad \bar{S}_i^{\phi\eta_i} = \bar{O}(h) \quad \bar{S}_i^{\eta_i \eta_i} = f_i(x) + \bar{O}(h).$$

Proof. The proof follows from standard calculations for kernel density estimators. Therefore, it is omitted. □

Lemma 3. *The following representation holds under Assumptions (A1) to (A6):*

For a boundary point x, i.e., $x = ch$ for some $c > 0$ in the support of $K(\cdot)$, we have

$$\hat{\phi} - \phi_0 - h^2 B^{(1)} = -\frac{\mathcal{C}^{-1}}{\bar{f}_N(0)} \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T [\mathbf{1}\{u_{it} \le 0\} - \tau]\left[\frac{X_{it} - x}{h} - \frac{\mathcal{C}_1}{c_0}\right] \frac{K_{it}}{h^d} + o_P(1/\sqrt{NTh^d}).$$

For a interior point x, we have

$$\hat{\phi} - \phi_0 = -\frac{\mathcal{K}_1^{-1}}{\bar{f}_N(x)} \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T [\mathbf{1}\{u_{it} \le 0\} - \tau]\frac{X_{it} - x}{h} \frac{K_{it}}{h^d} + o_P(1/\sqrt{NTh^d}).$$

Proof. We only provide the proof for the boundary point, which is more involved. Let $\{\delta_{NT}\}$ be a non-increasing sequence such that $\max_{i \le N} |\hat{\eta}_i - \eta_{0i}| = O_P(\delta_{NT})$, and let $\{\gamma_{NT}\}$ be a non-increasing sequence such that $\max_{i \le N} |\hat{\eta}_i - \eta_{0i}| \vee \|\hat{\phi} - \phi_0\| = O_P(\gamma_{NT})$.

Step 1 (Expansion):

Expanding $\bar{S}_i^{\eta_i}(\hat{\theta}_i)$ and $\bar{S}_i^{\phi}(\hat{\theta}_i)$ around θ_{0i} gives:

$$\bar{S}_i^{\phi}(\hat{\theta}_i) = \bar{S}_i^{\phi\phi}(\hat{\phi} - \phi_0) + \bar{S}_i^{\phi\eta_i}(\hat{\eta}_i - \eta_{0i}) + \bar{O}_P(\gamma_{NT}) \cdot \|\hat{\phi} - \phi_0\| + \bar{O}_P(\delta_{NT}^2),$$
$$\bar{S}_i^{\eta_i}(\hat{\theta}_i) = (\bar{S}_i^{\eta_i \phi})'(\hat{\phi} - \phi_0) + \bar{S}_i^{\eta_i \eta_i}(\hat{\eta}_i - \eta_{0i}) + \bar{O}_P(\gamma_{NT}) \cdot \|\hat{\phi} - \phi_0\| + \bar{O}_P(\delta_{NT}^2).$$

Plugging the second equation into the first one, and using the results of Lemma 1 and Lemma 2 gives

$$[f_i(0)(\mathcal{C}_2 - \mathcal{C}_1\mathcal{C}_1'/c_0) + \bar{O}(h)] \cdot (\hat{\phi} - \phi_0) = \bar{S}_i^{\phi}(\hat{\theta}_i) - (\bar{S}_i^{\phi\eta_i}/\bar{S}_i^{\eta_i \eta_i}) \cdot \bar{S}_i^{\eta_i}(\hat{\theta}_i) + \bar{O}_P(\gamma_{NT})\|\hat{\phi} - \phi_0\| + \bar{O}_P(\delta_{NT}^2).$$

16

It follows that

$$[\bar{f}_N(0)(\mathcal{C}_2 - \mathcal{C}_1\mathcal{C}_1'/c_0)] \cdot (\hat{\phi} - \phi_0) = \frac{1}{N}\sum_{i=1}^{N}\bar{S}_i^{\phi}(\hat{\theta}_i) - \frac{1}{N}\sum_{i=1}^{N}(\bar{S}_i^{\phi\eta_i}/\bar{S}_i^{\eta_i\eta_i}) \cdot \bar{S}_i^{\eta_i}(\hat{\theta}_i)$$
$$+ O_P(h + \gamma_{NT})\|\hat{\phi} - \phi_0\| + O_P(\delta_{NT}^2). \quad \text{(A.6)}$$

Similarly, for each i we can obtain:

$$[f_i(0)c_0 + \bar{O}(h)] \cdot (\hat{\eta}_i - \eta_{0i}) = \bar{S}_i^{\eta_i}(\hat{\theta}_i) - f_i(0)\mathcal{C}_1'(\hat{\phi} - \phi_0) + \bar{O}_P(h + \gamma_{NT})\|\hat{\phi} - \phi_0\| + \bar{O}_P(\delta_{NT}^2). \quad \text{(A.7)}$$

Step 2 ($N^{-1}\sum_{i=1}^{N}\bar{S}_i^{\phi}(\hat{\theta}_i)$ and $N^{-1}\sum_{i=1}^{N}\bar{S}_i^{\eta_i}(\hat{\theta}_i)$):

First consider $N^{-1}\sum_{i=1}^{N}\bar{S}_i^{\phi}(\hat{\theta}_i)$. Define $Z_{it} = (X_{it} - x)/h \cdot K_{it}/h^d$. We can write

$$\frac{1}{N}\sum_{i=1}^{N}\bar{S}_i^{\phi}(\hat{\theta}_i) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\mathbf{1}\{Y_{it} \leq \hat{\theta}_i'W_{it}\} - \tau]Z_{it}$$
$$- \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\{[\mathbf{1}\{u_{it} \leq (\hat{\theta}_i - \theta_{0i})'W_{it}\} - \tau]Z_{it} - \mathbb{E}\left[[\mathbf{1}\{u_{it} \leq (\hat{\theta}_i - \theta_{0i})'W_{it}\} - \tau]Z_{it}\right]\right\}$$
$$+ \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\mathbf{1}\{u_{it} \leq (\hat{\theta}_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{Y_{it} \leq \hat{\theta}_i'W_{it}\}\right]Z_{it}.$$

Define

$$A_{NT}(\theta) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\{[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \tau]Z_{it} - \mathbb{E}\left[[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \tau]Z_{it}\right]\right\}$$

and

$$B_{NT}(\theta) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{Y_{it} \leq \theta_i'W_{it}\}\right]Z_{it},$$

then adding and subtracting terms, we can write

$$\frac{1}{N}\sum_{i=1}^{N}\bar{S}_i^{\phi}(\hat{\theta}_i) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\mathbf{1}\{Y_{it} \leq \hat{\theta}_i'W_{it}\} - \tau]Z_{it} - A_{NT}(\theta_0) + (A_{NT}(\theta_0) - A_{NT}(\hat{\theta})) + B_{NT}(\hat{\theta}).$$
$$\text{(A.8)}$$

Moreover, define

$$B_{NT}^{(1)}(\theta) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left\{ \left[\mathbf{1}\{u_{it} \le (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{Y_{it} \le \theta_i'W_{it}\} \right] Z_{it} \right.$$
$$\left. - \mathbb{E}\left[\left[\mathbf{1}\{u_{it} \le (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{Y_{it} \le \theta_i'W_{it}\} \right] Z_{it} \right] \right\}$$

$$B_{N}^{(2)}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\left[\mathbf{1}\{u_{it} \le (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{Y_{it} \le \theta_i'W_{it}\} \right] Z_{it}, \right]$$

we can write:
$$B_{NT}(\hat{\theta}) = B_{NT}^{(1)}(\theta_0) + (B_{NT}^{(1)}(\hat{\theta}) - B_{NT}^{(1)}(\theta_0)) + B_{N}^{(2)}(\hat{\theta}).$$

First, by the computational property of quantile regressions, the first term on the right-hand side of (A.8) is $O_P(T^{-1}h^{-d-1})$.

Second, consider $A_{NT}(\theta_0) - A_{NT}(\hat{\theta})$. Following the proof of Kato et al. (2012), we will show that

$$A_{NT}(\theta_0) - A_{NT}(\hat{\theta}) = O_P(\xi_{NT}) \quad \text{where} \quad \xi_{NT} = \frac{|\log(h^{d+2}\gamma_{NT})|}{Th^{d+1}} \vee \frac{\sqrt{\gamma_{NT}} \cdot \sqrt{|\log(h^{d+2}\gamma_{NT})|}}{\sqrt{Th^d}}. \tag{A.9}$$

Note that

$$|A_{NT}(\theta_0) - A_{NT}(\hat{\theta})| \le \frac{1}{Nh^{d+1}} \sum_{i=1}^{N} \sup_{|\eta_i - \eta_{0i}|, |\phi - \phi_0| \le \gamma_{NT}} \left| \frac{1}{T} \sum_{t=1}^{T} [l(\eta_i, \phi, V_{it}) - \mathbb{E}l(\eta_i, \phi, V_{it})] \right|,$$

where $V_{it} = [Y_{it}, X_{it}]$, and $l(\eta_i, \phi, V_{it}) = [\mathbf{1}\{u_{it} \le (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{u_{it} \le 0\}]Z_{it}h^{d+1}$. Define $\mathcal{L}_\gamma = \{l(\eta_i, \phi, V_{it}) : |\eta_i - \eta_{0i}| \vee \|\phi - \phi_0\| \le \gamma\}$, then the above inequality can be written as

$$|A_{NT}(\theta_0) - A_{NT}(\hat{\theta})| \le \frac{1}{Nh^{d+1}} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{t=1}^{T} l(V_{it}) - \mathbb{E}l(V_{it}) \right\|_{\mathcal{L}_{\gamma_{NT}}}.$$

Thus, to prove (A.9), it suffices to show that

$$\max_{i \le N} \mathbb{E} \left\| \frac{1}{T} \sum_{t=1}^{T} l(V_{it}) - \mathbb{E}l(V_{it}) \right\|_{\mathcal{L}_{\gamma_{NT}}} = O_P \left(\frac{|\log(h^{d+2}\gamma_{NT})|}{T} \vee \frac{\sqrt{h^{d+2}\gamma_{NT}} \cdot \sqrt{|\log(h^{d+2}\gamma_{NT})|}}{\sqrt{T}} \right). \tag{A.10}$$

Since $Z_{it}h^{d+1} = K_{it} \cdot (X_{it} - x)$ is uniformly bounded, the class of functions $\mathcal{L}_\infty = \{l(\eta_i, \phi, V_{it}) : \phi, \eta_i \in \mathbb{R}\}$ is a VC subgraph class, and $\mathbb{E}[l(\eta_i, \phi, V_{it})^2] = O(\gamma_{NT}h^{d+2})$ for $l(\eta_i, \phi, V_{it}) \in \mathcal{L}_{\gamma_{NT}}$, (A.10) follows from Proposition B.1 of Kato et al. (2012). Similarly, we can show that

$$B_{NT}^{(1)}(\hat{\theta}) - B_{NT}^{(1)}(\theta_0) = O_P(\xi_{NT}). \tag{A.11}$$

18

Third, consider $B_{NT}^{(1)}(\theta_0)$. Note that

$$B_{NT}^{(1)}(\theta_0) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left\{ \left[\mathbf{1}\{u_{it} \leq 0\} - \mathbf{1}\{u_{it} \leq -m(X_{it})\} \right] Z_{it} - \mathbb{E}\left[\left[\mathbf{1}\{u_{it} \leq 0\} - \mathbf{1}\{u_{it} \leq -m(X_{it})\} \right] Z_{it} \right] \right\}$$

where $Y_{it} = \theta_0' W_{it} + 0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R(X_{it}, x) + u_{it}$ and $m(X_{it}) = 0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R(X_{it}, x)$. Since

$$\mathbb{E}\left\| B_{NT}^{(1)}(\theta_0) \right\|^2 \leq \frac{1}{N^2 T} \sum_{i=1}^{N} \mathbb{E}\left[|\mathbf{1}\{u_{it} \leq 0\} - \mathbf{1}\{u_{it} \leq -m(X_{it})\}| \cdot \|Z_{it}\|^2 \right] = O(h^{2-d}/(NT)),$$

it follows that

$$B_{NT}^{(1)}(\theta_0) = o_P(1/\sqrt{NTh^d}). \tag{A.12}$$

Finally, consider $B_N^{(2)}(\hat{\theta})$. Write

$$B_N^{(2)}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\left[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it} - m(X_{it})\} \right] Z_{it} \right],$$

since for some $c \in (0,1)$ we have

$$\mathbb{E}\left[\left[\mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it}\} - \mathbf{1}\{u_{it} \leq (\theta_i - \theta_{0i})'W_{it} - m(X_{it})\} \right] Z_{it} \right]$$
$$= \mathbb{E}\left[f_{u,i}((\theta_i - \theta_{0i})'W_{it} - c \cdot m(X_{it})|X_{it})m(X_{it})Z_{it} \right]$$
$$= \mathbb{E}\left[f_{u,i}(0|X_{it})m(X_{it})Z_{it} \right] + \mathbb{E}\left[[f_{u,i}((\theta_i - \theta_{0i})'W_{it} - c \cdot m(X_{it})|X_{it}) - f_{u,i}(0|X_{it})]m(X_{it})Z_{it} \right]$$

where the first term of the last expression is $0.5 f_i(0)h^2 \int_{\mathcal{B}} u'\ddot{q}_\tau(0)uuK(u)du + \bar{O}(h^3)$, and the second term of the last expression is bounded by

$$C\|\theta_i - \theta_{0i}\| \cdot \mathbb{E}[\|W_{it}\| \cdot |m(X_{it})| \cdot \|Z_{it}\|] + C\mathbb{E}[|m(X_{it})|^2\|Z_{it}\|] = \|\theta_i - \theta_{0i}\| \cdot \bar{O}(h^2) + \bar{O}(h^4).$$

It then follows that

$$B_N^{(2)}(\hat{\theta}) = 0.5h^2 \bar{f}_N(0) \int_{\mathcal{B}} u'\ddot{q}_\tau(0)uuK(u)du + O_P(\gamma_{NT} \cdot h^2) + O_P(h^3). \tag{A.13}$$

Combining results (A.8), (A.9), (A.11), (A.12) and (A.13) gives

$$\frac{1}{N} \sum_{i=1}^{N} \bar{S}_i^\phi(\hat{\theta}_i) = -\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} [\mathbf{1}\{u_{it} \leq 0\} - \tau]Z_{it} + 0.5h^2 \bar{f}_N(0) \int_{\mathcal{B}} u'\ddot{q}_\tau(0)uuK(u)du$$
$$+ O_P(1/(Th^{d+1})) + O_P(\xi_{NT}) + o_P(1/\sqrt{NTh^d}) + O_P(\gamma_{NT} \cdot h^2) + O_P(h^3). \tag{A.14}$$

19

Similarly, we can show that

$$
\frac{1}{N}\sum_{i=1}^{N}(\bar{S}_i^{\phi\eta_i}/\bar{S}_i^{\eta_i\eta_i})\cdot\bar{S}_i^{\eta_i}(\hat{\theta}_i)=-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\bar{S}_i^{\phi\eta_i}/\bar{S}_i^{\eta_i\eta_i})\cdot[\mathbf{1}\{u_{it}\le 0\}-\tau]K_{it}/h^d
$$

$$
+0.5h^2\bar{f}_N(0)\mathcal{C}_1/c_0\cdot\int_{\mathcal{B}}u'\ddot{q}_\tau(0)uK(u)du+O_P(1/(Th^{d+1}))
$$

$$
+O_P(\xi_{NT})+o_P(1/\sqrt{NTh^d})+O_P(\gamma_{NT}h^2)+O_P(h^3). \quad (A.15)
$$

Step 3 (Rate of convergence):

Plugging (A.14) and (A.15) into (A.6) and using $\bar{S}_i^{\phi\eta_i}/\bar{S}_i^{\eta_i\eta_i}=\mathcal{C}_1/c_0+\bar{O}(h)$ give

$$
[\bar{f}_N(0)(\mathcal{C}_2-\mathcal{C}_1\mathcal{C}_1'/c_0)]\cdot(\hat{\phi}-\phi_0)
$$

$$
=-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\mathbf{1}\{u_{it}\le 0\}-\tau]\left[\frac{X_{it}-x}{h}-\frac{\mathcal{C}_1}{c_0}\right]\frac{K_{it}}{h^d}+0.5h^2\bar{f}_N(0)\int_{\mathcal{B}}u'\ddot{q}_\tau(0)u\left(u-\frac{\mathcal{C}_1}{c_0}\right)K(u)du
$$

$$
+o_P\left(\|\hat{\phi}-\phi_0\|\right)+O_P(\delta_{NT}^2)+O_P(1/(Th^{d+1}))+O_P(\xi_{NT})+o_P(1/\sqrt{NTh^d})+O_P(h^3)+O_P(\gamma_{NT}h^2).
$$

$$(A.16)$$

It then follows from our assumptions that

$$
\hat{\phi}-\phi_0=o_P(1/\sqrt{Th^d})+O_P(\delta_{NT}^2)+O_P(h^2). \quad (A.17)
$$

Next, the above inequality and (A.7) imply that for some $C>0$,

$$
\max_{i\le N}|\hat{\eta}_i-\eta_{0i}|\le C\max_{i\le N}|\bar{S}_i^{\eta_i}(\hat{\theta}_i)|+o_P(1/\sqrt{Th^d})+O_P(h^2).
$$

Similar to the proof of Step 2, we can show that

$$
\bar{S}_i^{\eta_i}(\hat{\theta}_i)=\bar{O}_P(T^{-1}h^{-d-1})+\bar{O}_P(h^2)+\bar{o}_P(1/\sqrt{Th^d})-D_T(\theta_{0i})+(D_T(\theta_{0i})-D_T(\hat{\theta}_i))+G_T(\hat{\theta}_i)-G_T(\theta_{0i})
$$

where

$$
D_T(\theta_i)=\frac{1}{T}\sum_{t=1}^{T}\left\{[\mathbf{1}\{u_{it}\le(\theta_i-\theta_{0i})'W_{it}\}-\tau]K_{it}/h^d-\mathbb{E}\left[[\mathbf{1}\{u_{it}\le(\theta_i-\theta_{0i})'W_{it}\}-\tau]K_{it}/h^d\right]\right\},
$$

$$
G_T(\theta_i)=\frac{1}{T}\sum_{t=1}^{T}\left\{\left[\mathbf{1}\{u_{it}\le(\theta_i-\theta_{0i})'W_{it}\}-\mathbf{1}\{Y_{it}\le\theta_i'W_{it}\}\right]K_{it}/h^d\right.
$$

$$
\left.-\mathbb{E}\left[\left[\mathbf{1}\{u_{it}\le(\theta_i-\theta_{0i})'W_{it}\}-\mathbf{1}\{Y_{it}\le\theta_i'W_{it}\}\right]K_{it}/h^d\right]\right\}.
$$

Applying Lemma 2.2.9 and Lemma 2.2.10 of Van der Vaart and Wellner (1996) we can show that

$$\max_{i \leq N} |D_T(\theta_{0i})| = O_P(\sqrt{\log N}/\sqrt{Th^d}).$$

Next, following the proof of Kato et al. (2012), we can show that:

$$\max_{i \leq N} |D_T(\hat{\theta}_i) - D_T(\theta_{0i})| = o_P(\sqrt{\log N}/\sqrt{Th^d}), \text{ and } \max_{i \leq N} |G_T(\hat{\theta}_i) - G_T(\theta_{0i})| = o_P(\sqrt{\log N}/\sqrt{Th^d})$$

It then follows that

$$\max_{i \leq N} |\bar{S}_i^{\eta_i}(\hat{\theta}_i)| = O_P(\sqrt{\log N}/\sqrt{Th^d}) + O_P(h^2),$$

$$\max_{i \leq N} |\hat{\eta}_i - \eta_{0i}| = O_P(\sqrt{\log N}/\sqrt{Th^d}) + O_P(h^2), \tag{A.18}$$

$$\|\hat{\phi} - \phi_0\| = o_P(\sqrt{\log N}/\sqrt{Th^d}) + O_P(h^2). \tag{A.19}$$

Step 4 (Bahadur Representation):

By (A.18), (A.19) we have $\delta_{NT} = \gamma_{NT} = \sqrt{\log N}/\sqrt{Th^d}$. It then follows from (A.16) that

$$\hat{\phi} - \phi_0 - h^2 B^{(1)} = -\frac{\mathcal{C}^{-1}}{\bar{f}_N(0)} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} [\mathbf{1}\{u_{it} \leq 0\} - \tau] \left[\frac{X_{it} - x}{h} - \frac{\mathcal{C}_1}{c_0} \right] \frac{K_{it}}{h^d} + o_P \left(\|\hat{\phi} - \phi_0\| \right)$$

$$+ O_P(1/(Th^{d+1})) + O_P(\log N/Th^d) + O_P(\xi_{NT}) + o_P(1/\sqrt{NTh^d}) + O_P(h^3) + O_P(\gamma_{NT} h^2). \tag{A.20}$$

Assumption (A6) implies that $O_P(1/(Th^{d+1}))$, $O_P(\log N/Th^d)$, $O_P(\gamma_{NT} h^2)$ and $O_P(h^3)$ are all $o_P(\sqrt{NTh^d})$. Further, it can be shown that

$$O_P(\xi_{NT}) = O_P \left(\frac{(\log N)^{1/4} (\log T)^{1/2}}{(Th^d)^{3/4}} \right),$$

which is also $o_P(\sqrt{NTh^d})$ by Assumption (A6). So the desired result for the boundary point follows. The desired result for the interior points can be proved in the same way, by noting that for interior points $c_0 = 1, \mathcal{C}_1 = 0, \mathcal{C} = \mathcal{K}_1$ and $b = 0$. $\qquad\square$

Proof of Theorem 1:

Proof. By Lyapunov's CLT, we can show that:

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} [\mathbf{1}\{u_{it} \leq 0\} - \tau] \left[\frac{X_{it} - x}{h} - \frac{\mathcal{C}_1}{c_0} \right] \frac{K_{it}}{h^{d/2}} \xrightarrow{d} \mathcal{N}(0, \tau(1-\tau)\Omega^*), \tag{A.21}$$

where

$$\Omega^* = \left(\lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} f_{X,i}(0) \right) \cdot \int_{\mathcal{B}} (u - \mathcal{C}_1/c_0)(u - \mathcal{C}_1/c_0)' K^2(u) du$$

then the desired result for the boundary points follows from (A.21) and Lemma 3. The desired result for the interior points follows similarly. □

A.2 Proof of Theorem 2

We write $\check{\eta}_i, \check{\beta}$ instead of $\check{\eta}_{i,\tau}(x), \check{\beta}_\tau(x)$. Moreover, let $\check{\phi} = h\check{\beta}$, $\check{\theta}_i = (\check{\eta}_i, \check{\phi}')'$, $\check{\theta} = (\check{\eta}_1, \ldots, \check{\eta}_N, \check{\phi}')'$, and define $\varrho_\tau(u) = [\tau - G(u/b)]u$,

$$S_{T,i}^*(\theta_i) = \frac{1}{Th^d} \sum_{t=1}^{T} \varrho_\tau(Y_{it} - \theta_i' W_{it}) K_{it}, \quad \bar{S}_i^*(\theta_i) = \mathbb{E}[\varrho_\tau((\theta_{0i} - \theta_i)' W_{it} + u_{it}) K_{it}/h^d],$$

and $S_{NT}^*(\theta) = N^{-1} \sum_{i=1}^{N} S_{T,i}^*(\theta_i)$, $\bar{S}_N^*(\theta) = N^{-1} \sum_{i=1}^{N} \bar{S}_i^*(\theta_i)$.

We only prove the result for the boundary points. The proof for the interior points is almost the same.

Lemma 4. *Under Assumptions B1 to B4, we have* $\|\check{\phi} - \phi_0\|_1 = o_P(1)$ *and* $\max_{i \leq N} |\check{\eta}_i - \eta_{0i}| = o_P(1)$.

Proof. First consider $\check{\phi}$. By the definition of the estimators, there exists some $i \leq N$ such that $S_{T,i}^*(\check{\theta}_i) \leq S_{T,i}^*(\theta_{0i})$. For any $\delta > 0$, $B_i(\delta)$ is as defined in the proof of Lemma 1. Suppose that $\|\check{\phi} - \phi_0\|_1 > \delta$, then $\check{\theta}_i \in B_i^C(\delta)$. Similarly, by the convexity of $S_{T,i}$, we have

$$S_{T,i}(\tilde{\theta}_i) - S_{T,i}(\theta_{0i}) \leq \check{r}_i(S_{T,i}(\check{\theta}_i) - S_{T,i}(\theta_{0i})),$$

where $\tilde{\theta}_i = \check{r}_i \check{\theta}_i + (1 - \check{r}_i)\theta_{0i}$ is on the boundary of $B_i(\delta)$ and $\check{r}_i = \delta/(|\check{\eta}_i - \eta_{0i}| + \|\check{\phi} - \phi_0\|_1) < 1$. Adding and subtracting terms gives

$$\bar{S}_i(\tilde{\theta}_i) - \bar{S}_i(\theta_{0i}) \leq S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - [S_{T,i}(\tilde{\theta}_i) - \bar{S}_i(\tilde{\theta}_i)] + \check{r}_i(S_{T,i}(\check{\theta}_i) - S_{T,i}(\theta_{0i}) - S_{T,i}^*(\check{\theta}_i) + S_{T,i}^*(\theta_{0i})).$$

The last term on the right-hand side of the above inequality is $\bar{O}_P(b)$, because we can show that $\sup_{\theta_i} |S_{T,i}(\theta_i) - S_{T,i}^*(\theta_i)| \leq Cb \cdot T^{-1} \sum_{t=1}^{T} K_{it}/h^d$ for some $C < \infty$ (see Horowitz 1998), and it is easy to show that $\sup_{i \leq N} T^{-1} \sum_{t=1}^{T} K_{it}/h^d = O_P(1) + O_P(\sqrt{\log N/Th^d})$. The rest of the proof is similar to the proof of Lemma 1 given that $b \to 0$ as $N, T \to \infty$.

Now consider $\check{\eta}_i$. By definition of the estimators, we have $S_{T,i}^*(\check{\eta}_i, \check{\phi}) \leq S_{T,i}^*(\eta_{0i}, \check{\phi})$ for all

$i \leq N$. Therefore, if $|\check{\eta}_i - \eta_{0i}| > \delta$ for some $i \leq N$, then we have $\check{\theta}_i \in B_i^C(\delta)$, and

$$(S_{T,i}(\tilde{\theta}_i) - S_{T,i}(\theta_{0i}))/\check{r}_i \leq S_{T,i}(\check{\theta}_i) - S_{T,i}(\theta_{0i}).$$

where r_i and $\tilde{\theta}$ is as define above. Adding the subtracting terms, we can write

$$S_{T,i}(\check{\theta}_i) - S_{T,i}(\theta_{0i}) = S_{T,i}^*(\check{\theta}_i) - S_{T,i}^*(\eta_{0i}, \check{\phi}) + S_{T,i}^*(\eta_{0i}, \check{\phi}) - S_{T,i}^*(\theta_{0i}) + \left[S_{T,i}(\check{\theta}_i) - S_{T,i}^*(\check{\theta}_i) - S_{T,i}(\theta_{0i}) + S_{T,i}^*(\theta_{0i}) \right].$$

Thus, from $S_{T,i}^*(\check{\eta}_i, \check{\phi}) \leq S_{T,i}^*(\eta_{0i}, \check{\phi})$ we have

$$\begin{aligned}
\bar{S}_i(\tilde{\theta}_i) - \bar{S}_i(\theta_{0i}) &\leq S_{T,i}(\theta_{0i}) - \bar{S}_i(\theta_{0i}) - [S_{T,i}(\tilde{\theta}_i) - \bar{S}_i(\tilde{\theta}_i)] \\
&\quad + \check{r}_i \left(S_{T,i}^*(\eta_{0i}, \check{\phi}) - S_{T,i}^*(\theta_{0i}) + \left[S_{T,i}(\check{\theta}_i) - S_{T,i}^*(\check{\theta}_i) - S_{T,i}(\theta_{0i}) + S_{T,i}^*(\theta_{0i}) \right] \right).
\end{aligned}$$

Since $\|\check{\phi} - \phi_0\|_1 = o_P(1)$, the last term on the RHS of the above inequality is $\bar{o}_P(1) + \bar{O}_P(b)$. The rest of the proof is similar to the proof of Lemma 1. $\qquad\square$

Lemma 5. *Let $\mathcal{B}_{i,\delta}$ be a neighbourhood of θ_{0i}, then under Assumptions B1 to B4, we have*

$$\sup_{\theta_i \in \mathcal{B}_{i,\delta}} \|S_{T,i}^{*\phi\phi\phi_j}(\theta_i)\| = \bar{O}_P(1), \ \sup_{\theta_i \in \mathcal{B}_{i,\delta}} \|S_{T,i}^{*\eta_i\phi\phi_j}(\theta_i)\| = \bar{O}_P(1), \ \sup_{\theta_i \in \mathcal{B}_{i,\delta}} \|S_{T,i}^{*\phi\phi\eta_i}(\theta_i)\| = \bar{O}_P(1),$$

$$\sup_{\theta_i \in \mathcal{B}_{i,\delta}} \|S_{T,i}^{*\eta_i\phi\eta_i}(\theta_i)\| = \bar{O}_P(1), \ \sup_{\theta_i \in \mathcal{B}_{i,\delta}} \|S_{T,i}^{*\phi\eta_i\eta_i}(\theta_i)\| = \bar{O}_P(1), \ \sup_{\theta_i \in \mathcal{B}_{i,\delta}} |S_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i)| = \bar{O}_P(1),$$

$$\sup_{\theta_i \in \mathcal{B}_{i,\delta}} \|S_{T,i}^{*\phi\eta_i\phi_j}(\theta_i)\| = \bar{O}_P(1), \ \sup_{\theta_i \in \mathcal{B}_{i,\delta}} |S_{T,i}^{*\eta_i\eta_i\phi_j}(\theta_i)| = \bar{O}_P(1).$$

Proof. To save space, we only prove that $\sup_{\theta_i \in \mathcal{B}_{i,\delta}} |S_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i)| = \bar{O}_P(1)$. The proofs of the other results are similar. Define $\varrho_\tau^{(j)}(u) = \partial^j \varrho_\tau(u)/\partial u^j$ and $g^{(j)} = \partial^j g(u)/\partial u^j$. Then we can write

$$\begin{aligned}
S_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i) &= -\frac{1}{T} \sum_{t=1}^{T} \varrho_\tau^{(3)}(Y_{it} - \theta_i' W_{it}) K_{it}/h^d \\
&= -\frac{3}{T} \sum_{t=1}^{T} g^{(1)} \left(\frac{Y_{it} - \theta_i' W_{it}}{b} \right) \cdot \frac{1}{b^2} \cdot \frac{K_{it}}{h^d} - \frac{1}{T} \sum_{t=1}^{T} g^{(2)} \left(\frac{Y_{it} - \theta_i' W_{it}}{b} \right) \cdot \frac{Y_{it} - \theta_i' W_{it}}{b^3} \cdot \frac{K_{it}}{h^d}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\bar{S}_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i) = \mathbb{E}\left[S_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i) \right] &= -3\mathbb{E}\left[g^{(1)} \left(\frac{Y_{it} - \theta_i' W_{it}}{b} \right) \cdot \frac{1}{b^2} \cdot \frac{K_{it}}{h^d} \right] \\
&\quad - \mathbb{E}\left[g^{(2)} \left(\frac{Y_{it} - \theta_i' W_{it}}{b} \right) \cdot \frac{Y_{it} - \theta_i' W_{it}}{b^3} \cdot \frac{K_{it}}{h^d} \right].
\end{aligned}$$

23

First,

$$\mathbb{E}\left[g^{(1)}\left(\frac{Y_{it}-\theta_i'W_{it}}{b}\right)\cdot\frac{1}{b^2}\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ \mathbb{E}\left[\int g^{(1)}\left(\frac{u-(\theta_i-\theta_{0i})'W_{it}+m(X_{it})}{b}\right)\frac{1}{b^2}f_{u,i}(u|X_{it})du\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ -\mathbb{E}\left[\int g(v)f_{u,i}^{(1)}\left(vb+(\theta_i-\theta_{0i})'W_{it}-m(X_{it})|X_{it}\right)dv\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ -\mathbb{E}\left[f_{u,i}^{(1)}\left((\theta_i-\theta_{0i})'W_{it}-m(X_{it})|X_{it}\right)\cdot\frac{K_{it}}{h^d}\right]+\bar{O}(b^m)$$

$$=\ -\mathbb{E}\left[f_{u,i}^{(1)}\left((\theta_i-\theta_{0i})'W_{it}|X_{it}\right)\cdot\frac{K_{it}}{h^d}\right]+\bar{O}(b^m)+\bar{O}(h^2)=\bar{O}(1).$$

Second,

$$\mathbb{E}\left[g^{(2)}\left(\frac{Y_{it}-\theta_i'W_{it}}{b}\right)\cdot\frac{Y_{it}-\theta_i'W_{it}}{b^3}\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ \mathbb{E}\left[\int g^{(2)}\left(\frac{u-(\theta_i-\theta_{0i})'W_{it}+m(X_{it})}{b}\right)\frac{u-(\theta_i-\theta_{0i})'W_{it}+m(X_{it})}{b^3}f_{u,i}(u|X_{it})du\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ \mathbb{E}\left[\int g^{(2)}\left(v\right)\frac{v}{b}f_{u,i}\left(vb+(\theta_i-\theta_{0i})'W_{it}-m(X_{it})|X_{it}\right)dv\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ -\mathbb{E}\left[\int g^{(1)}\left(v\right)vf_{u,i}^{(1)}\left(vb+(\theta_i-\theta_{0i})'W_{it}-m(X_{it})|X_{it}\right)dv\cdot\frac{K_{it}}{h^d}\right]$$

$$-\mathbb{E}\left[\int g^{(1)}\left(v\right)1/bf_{u,i}\left(vb+(\theta_i-\theta_{0i})'W_{it}-m(X_{it})|X_{it}\right)dv\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ b\cdot\mathbb{E}\left[\int g\left(v\right)vf_{u,i}^{(2)}\left(vb+(\theta_i-\theta_{0i})'W_{it}-m(X_{it})|X_{it}\right)dv\cdot\frac{K_{it}}{h^d}\right]-2\mathbb{E}\left[g^{(1)}\left(\frac{Y_{it}-\theta_i'W_{it}}{b}\right)\cdot\frac{1}{b^2}\cdot\frac{K_{it}}{h^d}\right]$$

$$=\ \bar{O}(b^m)-2\mathbb{E}\left[g^{(1)}\left(\frac{Y_{it}-\theta_i'W_{it}}{b}\right)\cdot\frac{1}{b^2}\cdot\frac{K_{it}}{h^d}\right].$$

It then follows that

$$\bar{S}_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i)=-\mathbb{E}\left[g^{(1)}\left(\frac{Y_{it}-\theta_i'W_{it}}{b}\right)\cdot\frac{1}{b^2}\cdot\frac{K_{it}}{h^d}\right]+\bar{O}(b^m)+\bar{O}(h^2)$$

$$=\mathbb{E}\left[f_{u,i}^{(1)}\left((\theta_i-\theta_{0i})'W_{it}|X_{it}\right)\cdot\frac{K_{it}}{h^d}\right]+\bar{O}(b^m)+\bar{O}(h^2)=\bar{O}(1).\quad\text{(A.22)}$$

Third, consider

$$S_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i)-\bar{S}_{T,i}^{*\eta_i\eta_i\eta_i}(\theta_i)=-\frac{1}{T}\sum_{t=1}^{T}\left\{\varrho_\tau^{(3)}(Y_{it}-\theta_i'W_{it})K_{it}/h^d-\mathbb{E}\left[\varrho_\tau^{(3)}(Y_{it}-\theta_i'W_{it})K_{it}/h^d\right]\right\}.$$

24

Similar to Lemma B.2 of Galvao and Kato (2016), we can show that

$$\max_{1 \le i \le N} \sup_{\theta_i \in \mathcal{B}_{i,\delta}} \left\| S_{T,i}^{*\eta_i \eta_i \eta_i}(\theta_i) - \bar{S}_{T,i}^{*\eta_i \eta_i \eta_i}(\theta_i) \right\| = o_P \left(\frac{\log N}{\sqrt{Tb^3 h^d}} \right) \tag{A.23}$$

Finally, the desired result follows from (A.22), (A.23) and $\log N / \sqrt{Tb^3 h^d} \to 0$. $\qquad\square$

Lemma 6. *Under Assumptions B1 to B4, we have*

$$S_{T,i}^{*\eta_i \eta_i} = c_0 f_i(0) + \bar{o}_P(\log N / \sqrt{Th^d b}) + \bar{O}_P(h + b^m), \quad S_{T,i}^{*\phi \eta_i} = \mathcal{C}_1 f_i(0) + \bar{o}_P(\log N / \sqrt{Th^d b}) + \bar{O}_P(h + b^m)$$

$$S_{T,i}^{*\phi \phi} = \mathcal{C}_2 f_i(0) + \bar{o}_P(\log N / \sqrt{Th^d b}) + \bar{O}_P(h + b^m)$$

Proof. The proof is similar to the previous lemma, therefore it is omitted. $\qquad\square$

Lemma 7. *Under Assumptions B1 to B4, we have*

$$\| \check{\phi} - \phi_0 \| = O_P(1/\sqrt{NTh^d}) + O_P(h^2 + b^m) + o_P \left(\max_{i \le N} |\check{\eta}_i - \eta_{0i}| \right).$$

Proof. Expanding the first order conditions we have

$$0 = S_{NT}^{*\phi}(\check{\theta}) = S_{NT}^{*\phi} + S_{NT}^{*\phi\phi} \cdot (\check{\phi} - \phi_0) + N^{-1} \sum_{i=1}^{N} S_{T,i}^{*\phi \eta_i} \cdot (\check{\eta}_i - \eta_{0i}) + 0.5 \sum_{j=1}^{d} S_{NT}^{*\phi \phi \phi_j}(\bar{\theta})(\check{\phi} - \phi_0)(\check{\phi}_j - \phi_{0j}) +$$

$$0.5 N^{-1} \sum_{i=1}^{N} S_{T,i}^{*\phi \phi \eta_i}(\bar{\theta}_i)(\check{\phi} - \phi_0)(\check{\eta}_i - \eta_{0i}) + 0.5 N^{-1} \sum_{i=1}^{N} S_{T,i}^{*\phi \eta_i \eta_i}(\bar{\theta}_i)(\check{\eta}_i - \eta_{0i})^2 +$$

$$0.5 N^{-1} \sum_{j=1}^{d} \sum_{i=1}^{N} S_{T,i}^{*\phi \eta_i \phi_j}(\bar{\theta}_i)(\check{\eta}_i - \eta_{0i})(\check{\phi}_j - \phi_{0j}), \tag{A.24}$$

$$0 = S_{T,i}^{*\eta_i}(\check{\theta}_i) = S_{T,i}^{*\eta_i} + S_{T,i}^{*\eta_i \eta_i} \cdot (\check{\eta}_i - \eta_{0i}) + S_{T,i}^{*\eta_i \phi} \cdot (\check{\phi} - \phi_0) + 0.5 \sum_{j=1}^{d} S_{T,i}^{*\eta_i \phi \phi_j}(\bar{\theta}_i)(\check{\phi} - \phi_0)(\check{\phi}_j - \phi_{0j}) +$$

$$0.5 S_{T,i}^{*\eta_i \phi \eta_i}(\bar{\theta}_i)(\check{\phi} - \phi_0)(\check{\eta}_i - \eta_{0i}) + 0.5 S_{T,i}^{*\eta_i \eta_i \eta_i}(\bar{\theta}_i)(\check{\eta}_i - \eta_{0i})^2 + 0.5 \sum_{j=1}^{d} S_{T,i}^{*\eta_i \eta_i \phi_j}(\bar{\theta}_i)(\check{\eta}_i - \eta_{0i})(\check{\phi}_j - \phi_{0j}),$$

$$\tag{A.25}$$

where $\bar{\theta}_i$ is between θ_{0i} and $\check{\theta}_i$, and $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_N)$. It then follows from Lemma 4 to Lemma 6 and (A.24), (A.25) that

$$\mathcal{C}_2 \bar{f}_N(0) \cdot (\check{\phi} - \phi_0) = -S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{N} \sum_{i=1}^{N} f_i(0)(\check{\eta}_i - \eta_{0i}) + o_P(\|\check{\phi} - \phi_0\|) + o_P\left(\max_{i \leq N} |\check{\eta}_i - \eta_{0i}|\right). \quad (A.26)$$

$$c_0 f_i(0) \cdot (\check{\eta}_i - \eta_{0i}) = -S_{T,i}^{*\eta_i} - f_i(0)\mathcal{C}_1'(\check{\phi} - \phi_0) + o_P(\|\check{\phi} - \phi_0\|) + o_P\left(\max_{i \leq N} |\check{\eta}_i - \eta_{0i}|\right). \quad (A.27)$$

Plugging (A.26) into (A.27) gives

$$\mathcal{C}\bar{f}_N(0) \cdot (\check{\phi} - \phi_0) = -\left[S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{c_0} \frac{1}{N} \sum_{i=1}^{N} S_{T,i}^{*\eta_i}\right] + o_P(\|\check{\phi} - \phi_0\|) + o_P\left(\max_{i \leq N} |\check{\eta}_i - \eta_{0i}|\right). \quad (A.28)$$

Write $Z_{it}^* = ((X_{it} - x)/h - \mathcal{C}_1/c_0) \cdot K_{it}/h^d$, we have

$$S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{c_0} \frac{1}{N} \sum_{i=1}^{N} S_{T,i}^{*\eta_i} = -\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}^*,$$

where $\varrho_\tau^{(1)}(u) = \tau - G(u/b) + g(u/b)u/b$. We can write

$$S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{c_0} \frac{1}{N} \sum_{i=1}^{N} S_{T,i}^{*\eta_i} = -\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varrho_\tau^{(1)}(u_{it})Z_{it}^* - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} [\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it}) - \varrho_\tau^{(1)}(u_{it})]Z_{it}^*.$$

The first term on the right-hand side of the above equation is $O_P(1/\sqrt{NTh^d}) + O(b^m)$ by the proof of the next lemma. Next, we focus on the second term on the RHS of the above equation, which can be written as:

$$-\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varrho_\tau^{(2)}(u_{it}) \left(0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R_\tau(x, X_{it})\right) Z_{it}^*$$

$$-\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varrho_\tau^{(3)}[u_{it} + c0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + cR_\tau(x, X_{it})] \left[0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R_\tau(x, X_{it})\right]^2 Z_{it}^*,$$

$$(A.29)$$

where $c \in [0, 1]$ and we have used the identity: $Y_{it} = \theta_{0i}'W_{it} + 0.5(X_{it} - x)'\ddot{q}_\tau(x)(X_{it} - x) + R_\tau(x, X_{it}) + u_{it}$.

First,

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\varrho_{\tau}^{(2)}(u_{it})(X_{it}-x)'\ddot{q}_{\tau}(x)(X_{it}-x)Z_{it}^{*} = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[\varrho_{\tau}^{(2)}(u_{it})(X_{it}-x)'\ddot{q}_{\tau}(x)(X_{it}-x)Z_{it}^{*}]+$$

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[\varrho_{\tau}^{(2)}(u_{it})(X_{it}-x)'\ddot{q}_{\tau}(x)(X_{it}-x)Z_{it}^{*} - \mathbb{E}[\varrho_{\tau}^{(2)}(u_{it})(X_{it}-x)'\ddot{q}_{\tau}(x)(X_{it}-x)Z_{it}^{*}]\Big].$$

(A.30)

For the first term on the RHS of (A.30) we have:

$$\mathbb{E}\big[\varrho_{\tau}^{(2)}(u_{it})(X_{it}-x)'\ddot{q}_{\tau}(x)(X_{it}-x)[(X_{it}-x)/h - \mathcal{C}_1'/c_0]K_{it}/h^d\big]$$
$$= h^2 f_i(0)\int_{\mathcal{B}} u'\ddot{q}_{\tau}(0)u(u - \mathcal{C}_1/c_0)K(u)du + \bar{O}_P(h^3) = \bar{O}_P(h^2), \quad \text{(A.31)}$$

and the second term can be shown to be $O_P(h^2/\sqrt{NTh^db}) = o_P(1/\sqrt{NTh^d})$.

Second, we can show that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\varrho_{\tau}^{(2)}(u_{it})R_{\tau}(x, X_{it})Z_{it}^{*} = O_P(h^3) + o_P(1/\sqrt{NTh^d}).$$

Third, we can show in a similar way that the second term of (A.29) is $o_P(h^3)+O_P(h^4/\sqrt{NTh^db^3}) = o_P(h^3) + o_P(1/\sqrt{NTh^d})$.

Combining the results above, we have

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\varrho_{\tau}^{(1)}(Y_{it} - \theta_{0i}'W_{it}) - \varrho_{\tau}^{(1)}(u_{it})]Z_{it}^{*} = O_P(h^2) + o_P(1/\sqrt{NTh^d})$$

and

$$S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{c_0}\frac{1}{N}\sum_{i=1}^{N}S_{T,i}^{*\eta_i} = O_P(h^2 + b^m) + O_P(1/\sqrt{NTh^d}).$$

Then the desired result follows from (A.28). $\qquad\square$

Lemma 8. *Under Assumptions B1 to B4, we have*

$$\max_{i\le N}|\check{\eta}_i - \eta_{0i}| = O_P(\sqrt{\log N}/\sqrt{Th^d}).$$

Proof. From Lemma 7 and (A.27)

$$c_0 f_i(0)(\check{\eta}_i - \eta_{0i}) = -S_{T,i}^{*\eta_i} + \bar{O}_P(1/\sqrt{NTh^d}) + \bar{O}_P(h^2 + b^m) + \bar{o}_P\left(\max_{i \leq N} |\check{\eta}_i - \eta_{0i}|\right),$$

it then suffices to show that

$$\max_{i \leq N} |S_{T,i}^{*\eta_i}| = \max_{i \leq N} \left|\frac{1}{T}\sum_{t=1}^{T} \varrho_\tau^{(1)}(Y_{it} - \theta_{0i}' W_{it})K_{it}/h^d\right| = O_P(\sqrt{\log N}/\sqrt{Th^d}) + O_P(h^2 + b^m). \quad \text{(A.32)}$$

Write

$$\frac{1}{T}\sum_{t=1}^{T} \varrho_\tau^{(1)}(Y_{it} - \theta_{0i}' W_{it})K_{it}/h^d = \frac{1}{T}\sum_{t=1}^{T}\left\{\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}' W_{it})K_{it}/h^d - \mathbb{E}\left[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}' W_{it})K_{it}/h^d\right]\right\}$$

$$+ \left\{\mathbb{E}\left[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}' W_{it})K_{it}/h^d\right] - \mathbb{E}\left[\varrho_\tau^{(1)}(u_{it})K_{it}/h^d\right]\right\} + \mathbb{E}\left[\varrho_\tau^{(1)}(u_{it})K_{it}/h^d\right]$$

From Lemma 2.2.9 and Lemma 2.2.10 of Van der Vaart and Wellner (1996) it can shown that the first term on the RHS of the above equation is $\bar{O}_P(\sqrt{\log N}/\sqrt{Th^d})$, and similar to the proof of Lemma 3 we can show that the second term is $\bar{O}_P(h^2)$. For the last term on the RHS of the above equation, we have

$$\mathbb{E}\left[\varrho_\tau^{(1)}(u_{it})K_{it}/h^d\right]$$

$$= \tau\mathbb{E}\left[K_{it}/h^d\right] - \mathbb{E}\left[G(u_{it}/b)K_{it}/h^d\right] + \mathbb{E}\left[g(u_{it}/b)u_{it}/b \cdot K_{it}/h^d\right]$$

$$= \tau\mathbb{E}\left[K_{it}/h^d\right] - \mathbb{E}\left[\int G(u/b)f_{u,i}(u|X_{it})du \cdot K_{it}/h^d\right] + \mathbb{E}\left[\int g(u/b)u/b \cdot f_{u,i}(u|X_{it})du \cdot K_{it}/h^d\right]$$

$$= \tau\mathbb{E}\left[K_{it}/h^d\right] - \mathbb{E}\left[\int g(v)F_{u,i}(vb|X_{it})dv \cdot K_{it}/h^d\right] + b\mathbb{E}\left[\int g(v)v \cdot f_{u,i}(vb|X_{it})dv \cdot K_{it}/h^d\right]$$

$$= \tau\mathbb{E}\left[K_{it}/h^d\right] - \tau\mathbb{E}\left[K_{it}/h^d\right] + \bar{O}(b^m) = \bar{O}(b^m).$$

Then the desired result follows. $\qquad\square$

Lemma 9. *Under Assumptions B1 to B4, we have*

$$\bar{f}_N(0)\mathcal{C} \cdot (\check{\phi} - \phi_0) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \varrho_\tau^{(1)}(u_{it})\left[\frac{X_{it} - x}{h} - \mathcal{C}_1/c_0\right]\frac{K_{it}}{h^d} + h^2 B^{(1)} + O(h^3)$$

$$+ N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0)) + o_P(\|\check{\phi} - \phi_0\|) + o_P(1/\sqrt{NTh^d}).$$

Proof. Plugging (A.25) into (A.24) we get:

$$\bar{f}_N(0)\mathcal{C} \cdot (\check{\phi} - \phi_0) = -\left[S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{c_0 N}\sum_{i=1}^{N}S_{T,i}^{*\eta_i}\right] - N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0](\check{\eta}_i - \eta_{0i})$$

$$- 0.5N^{-1}\sum_{i=1}^{N}\left[S_{T,i}^{*\phi\eta_i\eta_i}(\bar{\theta}_i) - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i\eta_i}(\bar{\theta}_i)/c_0\right](\check{\eta}_i - \eta_{0i})^2 + o_P(\|\check{\phi} - \phi_0\|) \quad \text{(A.33)}$$

First, from the proof of Lemma 7 we have

$$S_{NT}^{*\phi} - \frac{\mathcal{C}_1}{c_0 N}\sum_{i=1}^{N}S_{T,i}^{*\eta_i} = -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\varrho_\tau^{(1)}(u_{it})\left[\frac{X_{it}-x}{h} - \mathcal{C}_1/c_0\right]\frac{K_{it}}{h^d}$$

$$- 0.5h^2\bar{f}_N(0)\int_{\mathcal{B}}u'\ddot{q}_\tau(0)u(u - \mathcal{C}_1/c_0)K(u)du + \bar{O}_P(h^3) + o_P(1/\sqrt{NTh^d}). \quad \text{(A.34)}$$

Second, from the proof of Lemma 5 we have

$$S_{T,i}^{*\phi\eta_i\eta_i}(\bar{\theta}_i) = f_{u,i}^{(1)}(0|0)f_{X,i}(0) \cdot \mathcal{C}_1 + \bar{O}_P\left(\|\bar{\theta}_i - \theta_{0i}\|\right) + \bar{O}_P(h + b^m) + \bar{o}_P\left(\frac{\log N}{\sqrt{Tb^3h^d}}\right),$$

$$S_{T,i}^{*\eta_i\eta_i\eta_i}(\bar{\theta}_i) = f_{u,i}^{(1)}(0|0)f_{X,i}(0) \cdot c_0 + \bar{O}_P\left(\|\bar{\theta}_i - \theta_{0i}\|\right) + \bar{O}_P(h + b^m) + \bar{o}_P\left(\frac{\log N}{\sqrt{Tb^3h^d}}\right).$$

It then follows that

$$S_{T,i}^{*\phi\eta_i\eta_i}(\bar{\theta}_i) - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i\eta_i}(\bar{\theta}_i)/c_0 = \bar{O}_P\left(\|\bar{\theta}_i - \theta_{0i}\|\right) + \bar{O}_P(h + b^m) + \bar{o}_P\left(\frac{\log N}{\sqrt{Tb^3h^d}}\right).$$

The above equation and Lemma 8 imply that

$$N^{-1}\sum_{i=1}^{N}\left[S_{T,i}^{*\phi\eta_i\eta_i}(\bar{\theta}_i) - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i\eta_i}(\bar{\theta}_i)/c_0\right](\check{\eta}_i - \eta_{0i})^2 = O_P\left(\max_{i\le N}|\check{\eta}_i - \eta_{0i}|^3\right)$$

$$+ O_P\left(\max_{i\le N}|\check{\eta}_i - \eta_{0i}|^2\right) \cdot O_P(h) + O_P\left(\max_{i\le N}|\check{\eta}_i - \eta_{0i}|^2\right) \cdot o_P\left(\frac{\log N}{\sqrt{Tb^3h^d}}\right)$$

$$= O_P\left(\frac{(\log N)^{3/2}}{(Th^d)^{3/2}}\right) + O_P\left(\frac{h\log N}{Th^d}\right) + o_P\left(\frac{\log N}{Th^d} \cdot \frac{\log N}{\sqrt{Th^db^3}}\right) = o_P(1/\sqrt{NTh^d}). \quad \text{(A.35)}$$

Third, by (A.25), Lemma 6 and Lemma 8 we have

$$c_0 f_i(0)(\check{\eta}_i - \eta_{0i}) = -S_{T,i}^{*\eta_i} - (S_{T,i}^{*\eta_i\eta_i} - c_0 f_i(0))(\check{\eta}_i - \eta_{0i}) + \bar{O}_P(\check{\phi} - \phi_0) + \bar{O}_P(\log N/(Th^d))$$

$$= -S_{T,i}^{*\eta_i} + \bar{O}_P(\check{\phi} - \phi_0) + \bar{o}_P\left(\frac{(\log N)^{3/2}}{Th^d\sqrt{b}}\right) + \bar{O}_P\left(\frac{h\sqrt{\log N}}{\sqrt{Th^d}}\right)$$

Thus, it follows from the above result and Lemma 6 that

$$N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0](\check{\eta}_i - \eta_{0i}) = -N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))$$

$$+ o_P(\|\check{\phi} - \phi_0\|) + o_P\left(\frac{(\log N)^{5/2}}{(Th^d)^{3/2}b}\right) + o_P\left(\frac{h(\log N)^{3/2}}{Th^d\sqrt{b}}\right) + O_P\left(\frac{h^2\sqrt{\log N}}{\sqrt{Th^d}}\right)$$

$$= -N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0)) + o_P(\|\check{\phi} - \phi_0\|) + o_P(\sqrt{NTh^d}). \quad \text{(A.36)}$$

Finally, the desired result follows from (A.33) to (A.36). □

Lemma 10. *We have*

$$N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0)) = -\frac{\tau - 1/2}{Th^d}(\mathcal{D}_1/c_0 - \mathcal{C}_1 d_0/c_0^2) + o_P((Th^d)^{-1}).$$

Proof. **Step 1:**

First, write

$$\mathbb{E}[S_{T,i}^{*\phi\eta_i}S_{T,i}^{*\eta_i}] = -\mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d\right)\right]$$

$$= -\frac{1}{Th^d}\mathbb{E}\left[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})[(X_{it} - x)/h]K_{it}^2/h^d\right]$$

$$- \frac{T-1}{T}\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}]\cdot\mathbb{E}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d].$$

Second, it can be shown that

$$\mathbb{E}\left[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})[(X_{it} - x)/h]K_{it}^2/h^d\right] = (\tau - 1/2)f_i(0)\int_{\mathcal{B}}uK^2(u)du + \bar{o}(1).$$

It then follows that

$$\mathbb{E}[S_{T,i}^{*\phi\eta_i}S_{T,i}^{*\eta_i}] = -\frac{1}{Th^d}(\tau - 1/2)f_i(0)\int_{\mathcal{B}}uK^2(u)du$$

$$- \frac{T-1}{T}\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}]\cdot\mathbb{E}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d] + \bar{o}((Th^d)^{-1}). \quad \text{(A.37)}$$

Similarly, we can show that

$$\mathbb{E}[S_{T,i}^{*\eta_i\eta_i} S_{T,i}^{*\eta_i}] = -\frac{1}{Th^d}(\tau - 1/2)f_i(0)\int_{\mathcal{B}} K^2(u)du$$
$$-\frac{T-1}{T}\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d] \cdot \mathbb{E}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d] + \bar{o}((Th^d)^{-1}). \quad \text{(A.38)}$$

Third, it follows from (A.37) and (A.38) that

$$\mathbb{E}\left[N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] = -\frac{\tau - 1/2}{Th^d}(\mathcal{D}_1/c_0 - \mathcal{C}_1 d_0/c_0^2) + o((Th^d)^{-1})$$
$$-\frac{T-1}{T}\frac{1}{N}\sum_{i=1}^{N}\left[\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}] - \mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d]\mathcal{C}_1/c_0\right] \cdot \mathbb{E}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d]/(c_0 f_i(0)),$$
$$\text{(A.39)}$$

and we can show that $\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}] = \mathcal{C}_1 f_i(0) + \bar{O}(h)$, $\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d] = c_0 f_i(0) + \bar{O}(h)$ and $\mathbb{E}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d] = \bar{O}(h^2)$. It then follows that

$$\left[\mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})Z_{it}] - \mathbb{E}[\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d]\mathcal{C}_1/c_0\right] \cdot \mathbb{E}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}/h^d] = \bar{O}(h^3) = \bar{o}((Th^d)^{-1}).$$
$$\text{(A.40)}$$

Finally, it follows from (A.39) and (A.40) that

$$\mathbb{E}\left[N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] = -\frac{\tau - 1/2}{Th^d}(\mathcal{D}_1/c_0 - \mathcal{C}_1 d_0/c_0^2) + o((Th^d)^{-1}).$$
$$\text{(A.41)}$$

Step 2: Now we will show that

$$\text{Var}\left[N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] = o(1/(Th^d)^2). \quad \text{(A.42)}$$

Then the desired result follows from (A.41) and (A.42).

Define $\tilde{S}_{T,i}^{*\phi\eta_i} = S_{T,i}^{*\phi\eta_i} - \mathbb{E}[S_{T,i}^{*\phi\eta_i}]$, and $\tilde{S}_{T,i}^{*\eta_i\eta_i} = S_{T,i}^{*\eta_i\eta_i} - \mathbb{E}[S_{T,i}^{*\eta_i\eta_i}]$, then we can write

$$N^{-1}\sum_{i=1}^{N}[S_{T,i}^{*\phi\eta_i} - \mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0)) = N^{-1}\sum_{i=1}^{N}[\tilde{S}_{T,i}^{*\phi\eta_i} - \mathcal{C}_1\tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))$$
$$+ N^{-1}\sum_{i=1}^{N}[\bar{S}_{T,i}^{*\phi\eta_i} - \mathcal{C}_1\bar{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0)),$$

31

and it follows that for any $\omega \in \mathbb{R}^d$,

$$\mathrm{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[S_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 S_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] \leq 2\mathrm{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[\tilde{S}_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 \tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right]$$
$$+ 2\mathrm{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[\bar{S}_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 \bar{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right].$$

First, we have

$$\mathrm{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[\tilde{S}_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 \tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right]$$
$$= N^{-2}\sum_{i=1}^{N}\mathrm{Var}\left\{\omega'[\tilde{S}_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 \tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}\right\}/(c_0 f_i(0))^2$$
$$\leq N^{-2}\sum_{i=1}^{N}\mathbb{E}\left\{\omega'[\tilde{S}_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 \tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}\right\}^2/(c_0 f_i(0))^2.$$

Note that

$$\mathbb{E}\left\{\omega'[\tilde{S}_{T,i}^{*\phi\eta_i}-\mathcal{C}_1 \tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}\right\}^2$$
$$= \mathbb{E}\left\{\left(\frac{1}{Th^d}\sum_{t=1}^{T}a_{it}\right)^2\left(\frac{1}{Th^d}\sum_{t=1}^{T}b_{it}\right)^2\right\} = \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{p=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}a_{is}b_{ip}b_{ih}]$$
$$= \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}^2 b_{ip}b_{ih}] + \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}^2 b_{ih}^2] + \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}^2 b_{it}b_{ih}]$$
$$+ \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\mathbb{E}[a_{it}^2 b_{it}^2] + \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}[a_{it}a_{is}b_{it}b_{is}],$$

where $a_{it} = \omega'\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})[(X_{it} - x)/h - \mathcal{C}_1/c_0]K_{it} - \mathbb{E}[\omega'\varrho_\tau^{(2)}(Y_{it} - \theta_{0i}'W_{it})[(X_{it} - x)/h - \mathcal{C}_1/c_0]K_{it}]$, and $b_{it} = \varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}$. It can be shown that

$$\frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}^2 b_{ip}b_{ih}] = \bar{O}(h^4/(Th^d b)), \quad \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}^2 b_{ih}^2] = \bar{O}(1/[(Th^d)^2 b]),$$

$$\frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{h=1}^{T}\mathbb{E}[a_{it}^2 b_{it}b_{ih}] = \bar{O}(h^2/[(Th^d)^2 b]),$$

$$\frac{1}{(Th^d)^4}\sum_{t=1}^{T}\mathbb{E}[a_{it}^2 b_{it}^2] = \bar{O}(1/[(Th^d)^3 b]), \quad \frac{1}{(Th^d)^4}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}[a_{it}a_{is}b_{it}b_{is}] = \bar{O}(1/[(Th^d)^2]).$$

32

Thus, we have

$$\text{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[\tilde{S}_{T,i}^{*\phi\eta_i} - \mathcal{C}_1\tilde{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] = o(1/(NTh^d)). \qquad \text{(A.43)}$$

Second, define $\zeta_i = \omega'[\bar{S}_{T,i}^{*\phi\eta_i} - \mathcal{C}_1\bar{S}_{T,i}^{*\eta_i\eta_i}/c_0]/(c_0 f_i(0))$, then we can write

$$\text{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[\bar{S}_{T,i}^{*\phi\eta_i} - \mathcal{C}_1\bar{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] = \frac{1}{(NTh^d)^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\text{Var}[\varrho_\tau^{(1)}(Y_{it}-\theta_{0i}'W_{it})K_{it}]\zeta_i^2.$$

Since $\zeta_i = \bar{o}(1)$ and $\text{Var}[\varrho_\tau^{(1)}(Y_{it} - \theta_{0i}'W_{it})K_{it}] = \bar{O}(h^d)$, it follows that

$$\text{Var}\left[N^{-1}\sum_{i=1}^{N}\omega'[\bar{S}_{T,i}^{*\phi\eta_i} - \mathcal{C}_1\bar{S}_{T,i}^{*\eta_i\eta_i}/c_0]S_{T,i}^{*\eta_i}/(c_0 f_i(0))\right] = o(1/(NTh^d)). \qquad \text{(A.44)}$$

Finally, (A.42) follows from (A.43) and (A.44), and this concludes the proof. $\qquad\square$

Proof of Theorem 2:

Proof. It follows from Lemma 10 and Lemma 11 that

$$\check{\phi} - \phi_0 = \frac{\mathcal{C}^{-1}}{\bar{f}_N(0)NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\varrho_\tau^{(1)}(u_{it})\left[\frac{X_{it}-x}{h} - \mathcal{C}_1/c_0\right]\frac{K_{it}}{h^d} + h^2 B^{(1)}$$
$$+ \frac{1}{Th^d}B^{(2)} + o_P(\|\check{\phi}-\phi_0\|) + o_P(1/(Th^d)) + O(h^3).$$

It then suffices to show that

$$\frac{\mathcal{C}^{-1}}{\bar{f}_N(0)\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varrho_\tau^{(1)}(u_{it})\left[\frac{X_{it}-x}{h} - \mathcal{C}_1/c_0\right]\frac{K_{it}}{\sqrt{h^d}} \xrightarrow{d} \mathcal{N}(0, \tau(1-\tau)\sigma(0)\Omega). \qquad \text{(A.45)}$$

First, we can show that

$$\mathbb{E}\left[\varrho_\tau^{(1)}(u_{it})\left[\frac{X_{it}-x}{h} - \mathcal{C}_1/c_0\right]\frac{K_{it}}{\sqrt{h^d}}\right] = \bar{O}(h^{d/2}b^m) = \bar{o}(1/\sqrt{NT}).$$

Second, we can show that

$$\text{Var}\left[\varrho_\tau^{(1)}(u_{it})\left[\frac{X_{it}-x}{h} - \mathcal{C}_1/c_0\right]\frac{K_{it}}{\sqrt{h^d}}\right] = \tau(1-\tau)f_{X,i}(0)\int_{\mathcal{B}}(u-\mathcal{C}_1/c_0)(u-\mathcal{C}_1/c_0)'K^2(u)du + \bar{o}(1),$$

33

then ([A.45](#)) follows from Lyapunov's CLT. $\qquad\qquad\square$

A.3 Tables

Table 1: Biases and MSEs of the Estimators at $\tau = 0.25$ with Gaussian Errors.

		Bias				MSE			
x	$\beta_\tau(x)$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$
-2.0	1.603	-0.762	0.156	-0.747	-0.092	0.934	0.727	0.866	0.485
-1.6	1.572	-0.179	-0.015	-0.189	-0.023	0.082	0.084	0.088	0.098
-1.2	1.518	-0.082	-0.031	-0.084	-0.028	0.021	0.023	0.022	0.026
-0.8	1.421	-0.077	-0.052	-0.071	-0.048	0.013	0.013	0.011	0.013
-0.4	1.251	0.061	-0.052	-0.061	-0.051	0.007	0.008	0.008	0.009
0.0	1.000	0.001	0.000	-0.005	-0.005	0.003	0.004	0.003	0.005
0.4	0.750	0.060	0.049	0.055	0.046	0.007	0.007	0.007	0.008
0.8	0.579	0.076	0.051	0.074	0.053	0.012	0.012	0.013	0.015
1.2	0.482	0.091	0.040	0.085	0.031	0.022	0.022	0.022	0.028
1.6	0.428	0.188	0.021	0.183	0.021	0.090	0.087	0.093	0.113
2.0	0.397	0.736	-0.175	0.730	0.086	0.872	0.679	0.814	0.464

Note: The DGP considered in this table is: $Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it}$, $X_{it} \sim i.i.d \, \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i, \epsilon_{it} \sim i.i.d \, \mathcal{N}(0,1)$, so $\beta_\tau(x) = 1 + \Phi^{-1}(\tau) \cdot x/\sqrt{1 + x^2}$.

Table 2: Biases and MSEs of the Estimators at $\tau = 0.25$ with $T(3)$ Errors.

		Bias				MSE			
x	$\beta_\tau(x)$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$
-2.0	1.684	-0.981	0.150	-0.915	-0.198	1.428	0.896	1.308	0.715
-1.6	1.649	-0.241	-0.026	-0.230	-0.007	0.130	0.125	0.129	0.144
-1.2	1.588	-0.112	-0.041	-0.117	-0.046	0.032	0.033	0.037	0.041
-0.8	1.478	-0.090	-0.057	-0.091	-0.058	0.018	0.018	0.018	0.019
-0.4	1.284	-0.069	-0.055	-0.069	-0.057	0.010	0.011	0.010	0.012
0.0	1.000	0.001	0.007	-0.004	-0.006	0.005	0.007	0.005	0.008
0.4	0.716	0.071	0.059	0.070	0.059	0.010	0.011	0.010	0.012
0.8	0.522	0.095	0.059	0.097	0.063	0.019	0.018	0.020	0.021
1.2	0.412	0.116	0.046	0.115	0.039	0.033	0.035	0.036	0.042
1.6	0.351	0.218	-0.001	0.219	0.004	0.125	0.128	0.129	0.151
2.0	0.316	0.895	-0.238	0.911	0.220	1.253	0.949	1.266	0.701

Note: The DGP considered in this table is: $Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it}$, $X_{it} \sim i.i.d \, \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i \sim i.i.d \, \mathcal{N}(0,1)$, $\epsilon_{it} \sim T(3)$, so $\beta_{0.25}(x) = 1 - 0.765 \cdot x/\sqrt{1 + x^2}$.

Table 3: Biases and MSEs of the Estimators at $\tau = 0.5$ with Gaussian Errors.

		Bias				MSE			
x	$\beta_\tau(x)$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$
-2.0	1.000	-0.002	0.007	-0.014	-0.009	0.236	0.455	0.259	0.493
-1.6	1.000	0.005	0.015	0.007	-0.020	0.041	0.070	0.043	0.079
-1.2	1.000	0.007	0.006	0.005	-0.006	0.012	0.018	0.013	0.022
-0.8	1.000	-0.003	-0.003	-0.005	-0.006	0.005	0.007	0.005	0.008
-0.4	1.000	-0.004	-0.003	-0.006	-0.007	0.003	0.005	0.003	0.005
0.0	1.000	-0.005	-0.006	-0.007	-0.009	0.003	0.004	0.003	0.004
0.4	1.000	-0.003	-0.004	-0.003	-0.004	0.003	0.004	0.003	0.004
0.8	1.000	0.003	0.005	0.002	0.003	0.006	0.008	0.006	0.009
1.2	1.000	0.002	0.002	0.001	0.003	0.012	0.017	0.012	0.020
1.6	1.000	-0.003	-0.001	-0.006	-0.012	0.047	0.077	0.047	0.081
2.0	1.000	0.002	0.039	0.001	0.032	0.256	0.447	0.273	0.472

Note: The DGP considered in this table is: $Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it}$, $X_{it} \sim i.i.d \, \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i, \epsilon_{it} \sim i.i.d \, \mathcal{N}(0,1)$, so $\beta_\tau(x) = 1 + \Phi^{-1}(\tau) \cdot x/\sqrt{1 + x^2}$.

Table 4: Biases and MSEs of the Estimators at $\tau = 0.5$ with $T(3)$ Errors.

		Bias				MSE			
x	$\beta_\tau(x)$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$
-2.0	1.000	-0.041	-0.032	0.007	-0.012	0.369	0.587	0.337	0.561
-1.6	1.000	-0.008	-0.007	0.011	0.015	0.050	0.079	0.052	0.088
-1.2	1.000	-0.005	-0.004	0.004	0.005	0.012	0.018	0.017	0.027
-0.8	1.000	-0.004	-0.002	0.001	0.002	0.006	0.009	0.007	0.010
-0.4	1.000	-0.003	-0.003	-0.000	0.000	0.004	0.005	0.004	0.006
0.0	1.000	0.002	0.002	-0.002	-0.003	0.003	0.004	0.003	0.005
0.4	1.000	0.002	0.003	0.002	-0.003	0.004	0.005	0.004	0.006
0.8	1.000	-0.005	-0.006	0.008	0.007	0.007	0.010	0.007	0.011
1.2	1.000	-0.032	-0.002	0.001	0.002	0.012	0.018	0.017	0.026
1.6	1.000	-0.001	-0.009	-0.012	-0.022	0.048	0.075	0.057	0.094
2.0	1.000	0.016	-0.007	-0.012	-0.009	0.342	0.565	0.354	0.580

Note: The DGP considered in this table is: $Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it}$, $X_{it} \sim i.i.d \, \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i \sim i.i.d \, \mathcal{N}(0,1)$, $\epsilon_{it} \sim T(3)$, so $\beta_{0.5}(x) = 1$.

Table 5: Biases and MSEs of the Estimators at $\tau = 0.75$ with Gaussian Errors.

		Bias				MSE			
x	$\beta_\tau(x)$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$
-2.0	0.397	0.744	-0.147	0.751	0.112	0.828	0.599	0.863	0.518
-1.6	0.428	0.194	-0.034	0.194	0.027	0.084	0.077	0.088	0.096
-1.2	0.482	0.087	0.033	0.089	0.041	0.022	0.024	0.022	0.028
-0.8	0.579	0.068	0.043	0.069	0.045	0.011	0.011	0.011	0.014
-0.4	0.750	0.054	0.042	0.054	0.044	0.006	0.007	0.007	0.008
0.0	1.000	-0.001	-0.001	-0.001	-0.001	0.003	0.004	0.003	0.005
0.4	1.251	-0.058	-0.049	-0.058	-0.048	0.007	0.008	0.008	0.009
0.8	1.421	-0.072	-0.048	-0.071	-0.046	0.011	0.011	0.011	0.012
1.2	1.518	-0.087	-0.035	-0.090	-0.041	0.022	0.022	0.023	0.027
1.6	1.572	-0.195	-0.030	-0.197	-0.024	0.083	0.071	0.087	0.085
2.0	1.603	-0.737	0.217	-0.737	-0.053	0.868	0.696	0.875	0.497

Note: The DGP considered in this table is: $Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it}$, $X_{it} \sim i.i.d \ \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i, \epsilon_{it} \sim i.i.d \ \mathcal{N}(0,1)$, so $\beta_\tau(x) = 1 + \Phi^{-1}(\tau) \cdot x/\sqrt{1 + x^2}$.

Table 6: Biases and MSEs of the Estimators at $\tau = 0.75$ with $T(3)$ Errors.

		Bias				MSE			
x	$\beta_\tau(x)$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$	$\hat{\beta}_\tau$	$\hat{\beta}_\tau^{bc}$	$\check{\beta}_\tau$	$\check{\beta}_\tau^{bc}$
-2.0	0.316	0.936	-0.222	0.960	0.264	1.311	0.932	1.408	0.832
-1.6	0.351	0.242	0.019	0.258	0.058	0.130	0.117	0.139	0.133
-1.2	0.412	0.110	0.037	0.120	0.044	0.032	0.033	0.036	0.040
-0.8	0.522	0.086	0.051	0.088	0.048	0.017	0.016	0.017	0.018
-0.4	0.716	0.065	0.049	0.066	0.050	0.009	0.010	0.010	0.012
0.0	1.000	-0.003	-0.006	-0.000	-0.001	0.005	0.007	0.005	0.008
0.4	1.294	-0.070	-0.057	-0.067	-0.055	0.010	0.011	0.010	0.012
0.8	1.478	-0.087	-0.052	-0.083	-0.047	0.016	0.015	0.016	0.017
1.2	1.588	-0.105	-0.034	-0.104	-0.038	0.032	0.033	0.034	0.041
1.6	1.649	-0.257	-0.059	-0.242	-0.030	0.144	0.135	0.136	0.147
2.0	1.684	-0.980	0.155	-0.985	-0.302	1.434	0.918	1.475	0.796

Note: The DGP considered in this table is: $Y_{it} = \beta X_{it} + \alpha_i + \sqrt{1 + X_{it}^2} \cdot \epsilon_{it}$, $X_{it} \sim i.i.d \ \mathcal{N}(0,1) \cdot \mathbf{1}\{|X_{it}| \leq 2\}$, $\alpha_i \sim i.i.d \ \mathcal{N}(0,1)$, $\epsilon_{it} \sim T(3)$, so $\beta_{0.75}(x) = 1 + 0.765 \cdot x/\sqrt{1 + x^2}$.

References

Altonji, J. G. and R. L. Matzkin (2005). Cross section and panel data estimators for nonseparable models with endogenous regressors. *Econometrica 73*(4), 1053–1102.

Amemiya, T. (1982). Two stage least absolute deviations estimators. *Econometrica*, 689–711.

Arellano, M. and S. Bonhomme (2011). Nonlinear panel data analysis. *Annual Review of Economics 3*(1), 395–424.

Arellano, M. and S. Bonhomme (2016). Nonlinear panel data estimation via quantile regressions. *The Econometrics Journal 19*(3), C61–C94.

Cai, Z., L. Chen, and Y. Fang (2018). A semiparametric quantile panel data model with an application to estimating the growth effect of fdi. *Journal of Econometrics 206*(2), 531–553.

Canay, I. A. (2011). A simple approach to quantile regression for panel data. *The Econometrics Journal 14*(3), 368–386.

Chaudhuri, P. et al. (1991). Nonparametric estimates of regression quantiles and their local bahadur representation. *The Annals of statistics 19*(2), 760–777.

Chernozhukov, V., I. Fernández-Val, J. Hahn, and W. Newey (2013). Average and quantile effects in nonseparable panel models. *Econometrica 81*(2), 535–580.

Chernozhukov, V., I. Fernandez-Val, S. Hoderlein, H. Holzmann, and W. Newey (2015). Nonparametric identification in panels using quantiles. *Journal of Econometrics 188*(2), 378–392.

Dhaene, G. and K. Jochmans (2015). Split-panel jackknife estimation of fixed-effect models. *The Review of Economic Studies 82*(3), 991–1030.

Evdokimov, K. (2010). Identification and estimation of a nonparametric panel data model with unobserved heterogeneity. *Department of Economics, Princeton University*.

Fan, J., T.-C. Hu, and Y. K. Truong (1994). Robust non-parametric function estimation. *Scandinavian Journal of Statistics*, 433–446.

Fernández-Val, I. and M. Weidner (2018). Fixed effects estimation of large-T panel data models. *Annual Review of Economics 10*, 109–138.

Galvao, A. F. (2011). Quantile regression for dynamic panel data with fixed effects. *Journal of Econometrics 164*(1), 142–157.

Galvao, A. F. and K. Kato (2016). Smoothed quantile regression for panel data. *Journal of Econometrics 193*(1), 92–112.

Galvao, A. F. and G. V. Montes-Rojas (2010). Penalized quantile regression for dynamic panel data. *Journal of Statistical Planning and Inference 140*(11), 3476–3497.

Graham, B. S., J. Hahn, A. Poirier, and J. L. Powell (2018). A quantile correlated random coefficients panel data model. *Journal of Econometrics 206*(2), 305–335.

Hahn, J. and W. Newey (2004). Jackknife and analytical bias reduction for nonlinear panel models. *Econometrica 72*(4), 1295–1319.

Honda, T. (2000). Nonparametric estimation of a conditional quantile for α-mixing processes. *Annals of the Institute of Statistical Mathematics 52*(3), 459–470.

Horowitz, J. L. (1998). Bootstrap methods for median regression models. *Econometrica*, 1327–1351.

Hu, Y. and S. M. Schennach (2008). Instrumental variable treatment of nonclassical measurement error models. *Econometrica 76*(1), 195–216.

Kato, K., A. F. Galvao, and G. V. Montes-Rojas (2012). Asymptotics for panel quantile regression models with individual effects. *Journal of Econometrics 170*(1), 76–91.

Koenker, R. (2004). Quantile regression for longitudinal data. *Journal of Multivariate Analysis 91*(1), 74–89.

Lamarche, C. (2010). Robust penalized quantile regression estimation for panel data. *Journal of Econometrics 157*(2), 396–408.

Lancaster, T. (2000). The incidental parameter problem since 1948. *Journal of Econometrics 95*(2), 391–413.

Qu, Z. and J. Yoon (2015). Nonparametric estimation and inference on conditional quantile processes. *Journal of Econometrics 185*(1), 1–19.

Su, L. and T. Hoshino (2016). Sieve instrumental variable quantile regression estimation of functional coefficient models. *Journal of Econometrics 191*(1), 231–254.

Su, L. and A. Ullah (2009). Nonparametric prewhitening estimators for conditional quantiles. *Statistica Sinica 18*, 1131.

Van der Vaart, A. and J. Wellner (1996). *Weak convergence and empirical processes*. Springer, New York.

Yan, K. and Q. Li (2018). Nonparametric estimation of a conditional quantile function in a fixed effects panel data model. *Journal of Risk and Financial Management 11*(3), 44.

Yu, K. and M. Jones (1998). Local linear quantile regression. *Journal of the American statistical Association 93*(441), 228–237.