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Abstract

This paper studies the estimation of characteristic-based quantile factor models. In these models, the factor loadings are unknown functions of observed individual characteristics while the idiosyncratic errors are subject to conditional quantile restrictions. We propose a three-stage estimation procedure which is easily implementable in practice and exhibits nice properties. The convergence rates and the limiting distributions of the estimated factors and loading functions are derived under general conditions. In addition, we develop a consistent estimator for the number of factors at each quantile.

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1 Introduction

As a generalization of the classical factor analysis, approximate factor models (AFM) were first proposed by Chamberlain and Rothschild (1983) to characterize the co-movement of a large number of asset returns. The estimation and inference theory of AFM and its generalizations were later developed by Stock and Watson (2002), Bai and Ng (2002), Bai (2003) and many others — see Fan, Li, and Liao (2021) for a recent review. In this literature, the factors are usually assumed to be unobserved and therefore they need to be estimated together with the factor loadings. As a result, factor models suffer from a generic identification problem since the latent factors and their loadings can only be identified up to a rotation matrix.

In parallel, there has been a growing literature in finance that tries to explain the cross-sectional co-movements of stock returns. In this setup, the factors are usually approximated using the differences between the returns of portfolios sorted by some observed characteristics, e.g. capitalization and book-to-market ratio. This popular approach, pioneered by Fama and French (1993), has been extended to include additional factors, such as *momentum, profitability* and *investment*, together with well-known Fama-French three factors (see Fama and French (2015)).

On one the hand, the *latent factors* approach relies on easily implementable estimation methods, such as the principal component analysis (PCA), but it is often criticized for the lack of interpretation of the estimated factors. On the other hand, the Fama-French approach is unambiguous about the meaning of the factors, but their method of constructing the factor proxies quickly becomes unreliable for typical sample sizes as the number of factors grows (see Connor and Linton (2007)). A setup that avoids both shortcomings is the characteristic-based factor models (CFM) proposed by Connor and Linton (2007), where the factor loadings are assumed to be unknown functions of some observed characteristics while the factors remain unobserved. In CFM, the latent factors can be easily estimated from the data even when the number of factors is not small, and their interpretation hinges on the corresponding observed characteristics.

In an influential paper, Connor et al. (2012) have proposed an estimation method for the factors and factor loading functions in CFM. Recently, Fan, Liao, and Wang (2016) have considered a more general framework where the number of factors is allowed to differ from the number of observed characteristics, and the factor loadings are not necessarily fully explained by those characteristics. They proposed a methodology called projected principal component analysis (PPCA) to estimate their model, and showed that their estimators have faster convergence rates than those of the conventional PCA estimators.

Our main goal here is to extend the analysis of Connor et al. (2012) and Fan et al. (2016) to a new class of factor models called characteristic-based quantile factor models (CQFM). Compared to CFM, the main defining feature of CQFM is that the idiosyncratic errors are subject to quantile restrictions instead of mean restrictions. Moreover, the latent factors, loading functions, and the number of factors are all allowed to vary across quantiles in CQFM, providing a more complete picture of how the distributions of many asset returns are driven by a few common risk factors.

Our main contribution is a new three-stage estimation method for CQFM that is easy to implement in practice. First, at each time period, the outcomes (e.g., asset returns) are projected onto the space of the observed characteristics using sieve quantile regressions. Second, the factors and factor loadings are estimated by PCA using the fitted value from the first step. Finally, the whole loading functions are obtained as their projections onto the sieve space. In addition, we also propose a new estimator for the number of factors at each quantile.

The rates of convergences and the limiting distributions of the estimated factors and factor loading functions are derived under very general conditions, and the estimator for the number of factors is also shown to be consistent. In particular, all our asymptotic results are obtained without any moment restrictions on the idiosyncratic errors, making our method an ideal tool for studying financial markets where the error distributions are known to have heavy tails. Moreover, we only require the number of cross section observations (denoted as n) to diverge in our asymptotic analysis, while the number of time series observations (denoted as T) can be either fixed or diverging.

Our estimation method is similar to the PPCA approach of Fan et al. (2016). The main difference is that in the first step we use sieve quantile regressions to project the observed outcomes, while Fan et al. (2016) uses sieve least square regressions. As a result, our estimators are more robust to heavy tails and outliers, while the consistency of the PPCA estimators requires much stronger moment restrictions on the error terms. However, it should be noted that the robustness of our estimators comes at two major costs: (i) the average convergence rate of the estimated factors is generally slower than the PPCA estimators unless we make the strong assumption that the idiosyncratic errors are independent of the observed characteristics; (ii) we have to assume that the factor loadings are fully explained by the observed characteristics.

Our paper is closely related to work of Ma, Linton, and Gao (2021), who also studied the estimation and inference of CQFM. However, there exist several notable differences. First, similar to the setup of Connor and Linton (2007) and Connor et al. (2012), Ma et al. (2021) assumes that the number of factors is known and it is equal to the number of observed characteristics. In contrast, we allow the number factors (which can vary across quantiles) to be different from the number of characteristics, and the number of factors at each quantile can be consistently estimated from the data. Second, even though the estimation method of Ma et al. (2021) also starts with sieve quantile regressions to obtain an initial estimator for the quantile loading functions, their subsequent steps are based on an iterative minimization algorithm that jointly estimates the factors and loadings, while our second and third steps are based on PCA that are easier to compute. Third, the asymptotic results of Ma et al. (2021) are obtained as $n, T \to \infty$, while all our results hold either T is fixed or $T \to \infty$ as $n \to \infty$. There are also some differences in the assumptions imposed in these two papers, which will be further discussed as we present the main theoretical results.

Finally, the CQFM studied in this paper and Ma et al. (2021) can be viewed as a special case of the quantile factor models (QFM) recently proposed by Chen, Dolado, and Gonzalo (2021). In QFM, there are no restrictions on the factor loadings except for a standard rank condition, while in CQFM the loadings are modeled as unknown functions of some observed characteristics. As discussed above, this extra assumption imposed in CQFM helps us to interpret the latent factors, but also comes at the risk misspecification. Another advantage of the CQFM setup is that the model can be consistently estimated even when T is fixed, while the quantile factor analysis estimators proposed by Chen et al. (2021) are consistent only when both n and T go to infinity.

The rest of the paper is organized as follows: Section 2 introduces the model and the estimators. The asymptotic properties of the proposed estimators are presented in Section 3, where we also introduce a consistent estimator for the number of factors at each quantile. Section 4 provides simulation results. In Section 5, the proposed estimators are applied to study the risk factors and their factor loadings of stock returns. Finally, Section 6 concludes.

Notations: For any matrix $\boldsymbol{C}$, $\|\boldsymbol{C}\|$ and $\|\boldsymbol{C}\|_S$ denote the Frobenius norm and the spectral norm of $\boldsymbol{C}$, $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of $\boldsymbol{C}$ when the all eigenvalues are real. $\boldsymbol{C} > 0$ means that $\boldsymbol{C}$ is a positive definite matrix. For two sequences of positive constants $\{a_1, \ldots, a_n, \ldots\}$ and $\{b_1, \ldots, b_n, \ldots\}$, $a_n \asymp b_n$ means that a_n/b_n is bounded below and above for all large n. The symbol $\lesssim$ means that the left side is bounded by a positive constant times the right side. For a random vector (Y, X), $\mathsf{Q}_\tau[Y|X = x]$ denotes the τ-quantile of Y given $X = x$.

2 The Model and The Estimators

2.1 The Model

For a panel of observed data $\{y_{it}\}_{1\le i\le n, 1\le t\le T}$, Chen, Dolado, and Gonzalo (2021) proposed the following quantile factor models (QFM):

$$y_{it} = \boldsymbol{\lambda}_i'(\tau)\boldsymbol{f}_t(\tau) + u_{it}(\tau), \quad \tau \in (0, 1), \tag{1}$$

where $\boldsymbol{\lambda}_i(\tau), \boldsymbol{f}_t(\tau) \in \mathbb{R}^R$ are quantile-dependent *unobserved* quantile factor loadings and quantile factors respectively, R is the number of factors at τ,[1] and $u_{it}(\tau)$ is the idiosyncratic error satisfying $\mathsf{Q}_\tau[u_{it}(\tau)|\boldsymbol{\lambda}_i(\tau), \boldsymbol{f}_t(\tau)] = 0$.

In this paper, we focus on the characteristic-based QFM first studied by Ma et al. (2021), which can be viewed as a special case of the QFM (1). In particular, assume that there exists a vector of *observed* characteristics $\boldsymbol{x}_i \in \mathbb{R}^D$ for individual i such that

$$\boldsymbol{\lambda}_i(\tau) = \boldsymbol{g}_\tau(\boldsymbol{x}_i), \tag{2}$$

where $\boldsymbol{g}_\tau(\cdot) : \mathbb{R}^D \mapsto \mathbb{R}^R$ is a vector of unknown functions for each τ. Following Fan et al. (2016), we assume that the rth element of $\boldsymbol{g}_\tau(\boldsymbol{x}_i)$ is given by

$$g_{\tau,r}(\boldsymbol{x}_i) = \sum_{d=1}^{D} g_{\tau,rd}(x_{id})$$

where $g_{\tau,r1}, \ldots, g_{\tau,rD}$ are unknown functions and x_{id} is the dth element of $\boldsymbol{x}_i$.

Let $\boldsymbol{Y}$ be the $n \times T$ matrix of y_{it}, $\boldsymbol{F}_\tau$ be the $T \times R$ matrix of $\boldsymbol{f}_t(\tau)$, $\boldsymbol{G}_\tau(\boldsymbol{X})$ be the $n \times R$ matrix of $\boldsymbol{g}_\tau(\boldsymbol{x}_i)$, $\boldsymbol{U}_\tau$ be the $n \times T$ matrix of $u_{it}(\tau)$. Then models (1) and (2) can be written in the matrix form:

$$\boldsymbol{Y} = \boldsymbol{G}_\tau(\boldsymbol{X})\boldsymbol{F}_\tau' + \boldsymbol{U}_\tau. \tag{3}$$

Note that our setup above is more general than the models of Connor et al. (2012) and Ma et al. (2021), who required that the dimension of characteristics is equal to the number of factors ($D = R$), and that each of the loading functions is linked to only one observed characteristic, i.e., $g_{\tau,r}(\boldsymbol{x}_i) = g_{\tau,r}(x_{ir})$ for $r = 1, \ldots, R$. For example, if y_{it} represents the income of firm i at time t and the factors represent the monetary shock and fiscal shock, then it would be more reasonable to allow the reaction of the firm's income to the macro shocks to depend on several firm characteristics, such as size, leverage, growth and so on. Moreover, to implement the estimation methods of Connor et al. (2012) and Ma et al. (2021), one needs to assume that the number of factors are known, while our methodology allows the number of factors to be consistently estimated from the data (see Section 3.3 below).

Compared to the semiparametric factor models considered by Fan et al. (2016), the most obvious difference here is that the idiosyncratic errors in our model are subject to conditional quantile restrictions instead of conditional mean restrictions. One one hand, as pointed out in Chen et al. (2021), the QFM framework allows us to recovery different factor structures (including the factors, the loadings and the number of factors) across different quantiles, even when the distributions of the idiosyncratic errors have heavy tails. These features make QFM

[1] We suppress the dependence of R on τ to simplify the notations.

an ideal tool for studying the co-movement of the financial market, where the correlation of the tail risks between different assets is a major object of interest. On the other hand, a notable extension of Fan et al. (2016) compared to Connor et al. (2012) is that the factor loadings are allowed to be functions of other unobserved random variables except the observed characteristics. Doing so in the context of QFM will be very challenging. To see this, assume that

$$\boldsymbol{\lambda}_i(\tau) = \boldsymbol{g}_\tau(\boldsymbol{x}_i) + \boldsymbol{\gamma}_i,$$

where $\boldsymbol{\gamma}_i$ is unobserved and independent of $\boldsymbol{x}_i$. Then model (1) can be written as

$$y_{it} = \boldsymbol{g}_\tau(\boldsymbol{x}_i)'\boldsymbol{f}_t(\tau) + \tilde{u}_{it}(\tau) \quad \text{ where } \quad \tilde{u}_{it}(\tau) = u_{it}(\tau) + \boldsymbol{\gamma}_i'\boldsymbol{f}_t(\tau).$$

The above model can be viewed as a characteristic-based QFM with *measurement errors*, where the new error terms $\tilde{u}_{it}(\tau)$ no longer satisfy the conditional quantile restrictions even $\boldsymbol{x}_i$ and $\boldsymbol{\gamma}_i$ are independent. In fact, even in standard quantile regressions, dealing with measurement errors is not a trivial issue (see Hausman, Liu, Luo, and Palmer (2021) for example). Thus, in this paper, we assume that the factor loadings are fully explained by the observed characteristics.

2.2 The Estimators

To simplify the notations, in the rest of the paper, we suppress the τ-subscripts in the model and write $\boldsymbol{g}(), \boldsymbol{G}(\cdot), \boldsymbol{F}, \boldsymbol{U}$ instead of $\boldsymbol{g}_\tau(\cdot), \boldsymbol{G}_\tau(\cdot), \boldsymbol{F}_\tau, \boldsymbol{U}_\tau$.

Write $\theta_{0t}(\boldsymbol{x}_i) = \boldsymbol{g}(\boldsymbol{x}_i)'\boldsymbol{f}_t = \sum_{r=1}^R g_r(\boldsymbol{x}_i) f_{tr} = \sum_{r=1}^R (\sum_{d=1}^D g_{rd}(x_{id})) f_{tr}$. Let Θ be a space of continuous functions such that $\theta_{0t} \in \Theta$ for all $t = 1, \ldots, T$, and let $\{\Theta_n\}$ be a sequence of sieve spaces to approximate Θ. In particular, consider the following finite dimensional linear spaces:

$$\Theta_n = \left\{ h : \mathcal{X} \mapsto \mathbb{R}, h(\boldsymbol{x}) = \sum_{d=1}^{D}\sum_{j=1}^{k_n} a_{jd}\phi_j(x_d) : (a_{11}, \ldots, a_{jd}, \ldots, a_{k_n D}) \in \mathbb{R}^{Dk_n} \right\},$$

where $\mathcal{X} \subset \mathbb{R}^D$ is the support of $\boldsymbol{x}_i$, and $\phi_1, \ldots, \phi_{k_n}$ is a set of continuous basis functions. Write

$$\underbrace{\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)}_{Dk_n \times 1} = [\phi_1(x_{i1}), \ldots, \phi_{k_n}(x_{i1}), \ldots \phi_1(x_{id}), \ldots, \phi_{k_n}(x_{id}), \ldots, \phi_1(x_{iD}), \ldots, \phi_{k_n}(x_{iD})]'.$$

Suppose that for $r = 1, \ldots, R$, there exists $\boldsymbol{b}_{01}, \ldots, \boldsymbol{b}_{0R} \in \mathbb{R}^{Dk_n}$ such that for some constant $\alpha > 0$,

$$\max_{1 \le r \le R} \sup_{\boldsymbol{x} \in \mathcal{X}} \left| g_r(\boldsymbol{x}) - \boldsymbol{b}_{0r}'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}) \right| = O(k_n^{-\alpha}). \tag{4}$$

Then for $\boldsymbol{B}_0 = (\boldsymbol{b}_{01}, \ldots, \boldsymbol{b}_{0R}) \in \mathbb{R}^{Dk_n \times R}$, $\boldsymbol{a}_{0t} = \boldsymbol{B}_0 \boldsymbol{f}_t$ and $\pi_n\theta_{0t}(\cdot) = \boldsymbol{a}_{0t}'\boldsymbol{\phi}_{k_n}(\cdot)$, we have $\pi_n\theta_{0t} \in$

Θ_n for all t and

$$\max_{1\le t\le T}\sup_{\boldsymbol{x}\in\mathcal{X}}|\pi_n\theta_{0t}(\boldsymbol{x})-\theta_{0t}(\boldsymbol{x})|=O(k_n^{-\alpha}). \tag{5}$$

Our estimation method consists of the following three steps.

Step 1: Obtain the sieve estimator of θ_{0t}. Let $\rho_\tau(u)=(\tau-\mathbf{1}\{u\le 0\})u$ be the check function, and define $l(\theta,y_{it},\boldsymbol{x}_i)=\rho_\tau(y_{it}-\theta(\boldsymbol{x}_i))-\rho_\tau(y_{it}-\theta_{0t}(\boldsymbol{x}_i))$, $L_n(\theta)=n^{-1}\sum_{i=1}^n l(\theta,y_{it},\boldsymbol{x}_i)$. Then the sieve estimator $\hat{\theta}_{nt}$ is defined by

$$L_n(\hat{\theta}_{nt})\le\inf_{\theta\in\Theta_n}L_n(\theta).$$

In practice, $\hat{\theta}_{nt}$ can be obtained by a simple parametric quantile regression as follows:

$$\hat{\boldsymbol{a}}_t=\arg\min_{\boldsymbol{a}\in\mathbb{R}^{Dk_n}}\sum_{i=1}^N\rho_\tau\left(y_{it}-\boldsymbol{a}'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\right)\quad\text{and}\quad\hat{\theta}_{nt}(\cdot)=\hat{\boldsymbol{a}}_t'\boldsymbol{\phi}_{k_n}(\cdot).$$

Step 2: Write $\hat{y}_{it}=\hat{\theta}_{nt}(\boldsymbol{x}_i)=\hat{\boldsymbol{a}}_t'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)$ and let $\hat{\boldsymbol{Y}}$ be the $n\times T$ matrix of $\hat{y}_{it}$. Then, the estimator of $\boldsymbol{F}$, denoted as $\hat{\boldsymbol{F}}$, is the matrix of eigenvectors (multiplied by $\sqrt{T}$) associated with R largest eigenvalues of the $T\times T$ matrix $\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}}$. Moreover, the estimator of the characteristics-based loading matrix $\boldsymbol{G}(\boldsymbol{X})$ is given by $\hat{\boldsymbol{G}}(\boldsymbol{X})=\hat{\boldsymbol{Y}}\hat{\boldsymbol{F}}/T$. It is well known that these estimators are minimizers of the objective function: $L_{nT}(\boldsymbol{G}(\boldsymbol{X}),\boldsymbol{F})=\|\hat{\boldsymbol{Y}}-\boldsymbol{G}(\boldsymbol{X})'\boldsymbol{F}\|^2$, subject to the normalizations that $\boldsymbol{F}'\boldsymbol{F}/T=\boldsymbol{I}_R$ and $\boldsymbol{G}(\boldsymbol{X})'\boldsymbol{G}(\boldsymbol{X})/n$ is diagonal (see Stock and Watson (2002)).

Step 3: Estimate the factor loading functions: $g_r(\cdot)$ for $r=1,\dots,R$. Let $\boldsymbol{A}_0=(\boldsymbol{a}_{01},\dots,\boldsymbol{a}_{0T})$ and $\hat{\boldsymbol{A}}=(\hat{\boldsymbol{a}}_{01},\dots,\hat{\boldsymbol{a}}_{0T})$. Intuitively, $\hat{\boldsymbol{A}}\approx\boldsymbol{A}_0=\boldsymbol{B}_0\boldsymbol{F}'\approx\boldsymbol{B}_0\hat{\boldsymbol{F}}'$, then $\boldsymbol{B}$ can be simply estimated by

$$\hat{\boldsymbol{B}}=\hat{\boldsymbol{A}}\hat{\boldsymbol{F}}/T. \tag{6}$$

The estimator of $\boldsymbol{g}(\boldsymbol{x})$ for any $\boldsymbol{x}\in\mathcal{X}$ is given by $\hat{\boldsymbol{g}}(\boldsymbol{x})'=\boldsymbol{\phi}_{k_n}(\boldsymbol{x})'\hat{\boldsymbol{B}}$.

Remark 1. *The main difference between our three-stage estimation method above and the PPCA method of Fan et al. (2016) is how we project $\boldsymbol{y}_t$ onto the space of $\boldsymbol{X}$ in the first step, i.e., how $\boldsymbol{a}_{01},\dots,\boldsymbol{a}_{0T}$ are estimated. The use of sieve quantile regressions instead of the least squares projections is a natural choice since the idiosyncratic errors in our model are subject to conditional quantile restrictions. When the distributions of the errors are symmetric around 0, our estimation method at $\tau=0.5$ can be viewed as a robust version of the PPCA since the consistency of our estimators does not rely on moment restrictions of the errors (see Theorem 1 below).*

Remark 2. *The estimation method proposed by Ma et al. (2021) chooses $\boldsymbol{B}$ and $\boldsymbol{F}$ iteratively* *to minimize the following object function:*

$$L_{nT}(\boldsymbol{B}, \boldsymbol{F}) = \sum_{i=1}^{n} \sum_{t=1}^{T} \rho_\tau(y_{it} - \boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)' \boldsymbol{B} \boldsymbol{f}_t),$$

while the quantile factor analysis (QFA) estimator of Chen et al. (2021) employed a similar approach that estimates the factor loadings $\boldsymbol{G}(\boldsymbol{X})$ and $\boldsymbol{F}$ jointly. Accordingly, these two approaches require both n and T go to infinity to establish the consistency of the estimators. In contrast, as will be shown in the next section, the consistency of our estimators can be established either when T is fixed or T goes to infinity along with n. Moreover, as discussed above, our specification for the quantile loading functions is a special case of the the QFM of Chen et al. (2021), but it is more general than the model considered by Ma et al. (2021).

3 Asymptotic Properties of the Estimators

In this section, we derive the rates of the convergence and the asymptotic distributions of the estimators proposed in the previous section. In the first two subsections, the number of factors is assumed to be known. In the last subsection we propose an estimator for R and prove its consistency.

As in Chen et al. (2021) and Ma et al. (2021), in the asymptotic analysis the quantile factors are treated as non-random constants. Then the conditional quantile restrictions on the idiosyncratic errors can be written as

$$P[u_{it} \leq 0 | \boldsymbol{x}_i = \boldsymbol{x}] = \tau \text{ for any } \boldsymbol{x} \in \mathcal{X}. \tag{7}$$

Alternatively, all the assumptions and results we will present below can be understood as conditional on the realizations of the factors.

Last but not least, it should be noted that all the results to be presented below hold either (i) T is fixed and $n \to \infty$, or (ii) $n, T \to \infty$. The first case is also called the *high-dimension-low-sample-size* setup in the statistics literature (see Shen, Shen, and Marron (2013) and Jung and Marron (2009)). One of the main insights of Fan et al. (2016) is that in the context of CFM, dimensionality is a blessing rather than a curse, thus their PPCA estimators are consistent even T is fixed. Our results below extend the finite-T-consistency results of Fan et al. (2016) to CQFM.

3.1 Rates of Convergence

Suppose that the observed data $\{y_{it}\}$ are generated by (3) and $\{u_{it}\}$ satisfy (7). Let

$$\varepsilon_n = \sqrt{k_n/n} \vee k_n^{-\alpha} \quad \text{and} \quad \varepsilon_{nT} = \sqrt{\ln T} \vee 1 \cdot \varepsilon_n.$$

For any $\theta_1, \theta_2 \in \Theta$, define the pseudo-metric $d(\theta_1, \theta_2) \equiv \sqrt{\mathbb{E}\left(\theta_1(\boldsymbol{x}_i) - \theta_2(\boldsymbol{x}_i)\right)^2}$. The following set of conditions are needed to establish the uniform rate of convergence of $\hat{\theta}_{n1}, \ldots, \hat{\theta}_{nT}$, which is a crucial result for proving the other theorems.

Assumption 1. *Let M be a generic bounded constant.*
(i) Define $\boldsymbol{z}_i = (u_{i1}, \ldots, u_{iT}, \boldsymbol{x}_i)$. Then $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ are i.i.d. The distributions of $(u_{i1}, \boldsymbol{x}_i), \ldots, (u_{iT}, \boldsymbol{x}_i)$ are identical.
(ii) Equation (4) holds for some $\alpha \geq 1$.
(iii) $\mathcal{X} \subset \mathbb{R}^D$ is bounded, and $\sup_{\theta \in \Theta} \sup_{\boldsymbol{x} \in \mathcal{X}} |\theta(\boldsymbol{x})| < M$. $\|\boldsymbol{F}_t\| < M$ for all $t = 1, \ldots, T$.
(iv) The conditional density of u_{it} given $\boldsymbol{x}_i = \boldsymbol{x}$, denoted as $\mathsf{f}(\cdot|\boldsymbol{x})$, satisfies: $0 < \inf_{\mathcal{X}} \mathsf{f}(0|\boldsymbol{x}) \leq \sup_{\mathcal{X}} \mathsf{f}(0|\boldsymbol{x}) < \infty$ and $\sup_{\mathcal{X}} |\mathsf{f}(z|\boldsymbol{x}) - \mathsf{f}(0|\boldsymbol{x})| \to 0$ as $|z| \to 0$.
(v) As $n \to \infty$, $k_n \to \infty$ and $\varepsilon_{nT} \to 0$.

Assumption 1(i) is stronger than those imposed in Fan et al. (2016) and Ma et al. (2021), but it can be relaxed to allow for weak cross-sectional dependence — see Remark 3 below for the details. Assumption 1(ii) is a general condition on the sieve approximations that can be easily verified using more primitive conditions. For example, it holds if Θ is an $\alpha-$smooth Hölder space (see Chen (2007) for more examples). Assumption 1(iii) and Assumption 1(iv) are standard in sieve quantile regressions while the last condition imposes very mild restrictions on the size of T in the case where T goes to infinity simultaneously with n.

Proposition 1. *If Assumption 1 holds, then either T is fixed or $T \to \infty$ as $n \to \infty$, we have $\max_{1 \leq t \leq T} d(\hat{\theta}_{nt}, \theta_{0t}) = O_P(\varepsilon_{nT})$.*

Remark 3. *The proof of Proposition 1 is based on Corollary 1 of Chen and Shen (1998). In particular, we show that*

$$P\left[\max_t d(\hat{\theta}_{nt}, \theta_{0t}) \geq C\varepsilon_{nT}\right] \leq \sum_{t=1}^{T} P\left[d(\hat{\theta}_{nt}, \theta_{0t}) \geq C\varepsilon_{nT}\right] \leq c_1 \exp\left\{C^2 \ln T(1 - c_2 n \varepsilon_n^2)\right\}$$

for any $C \geq 1$ and some constants c_1, c_2. Moreover, as shown in Chen and Shen (1998), the above inequality holds when the observations are generated from a stationary uniform (ϕ-) mixing sequence with $\phi(j) \lesssim j^{-\zeta}$ for some $\zeta > 1$. Thus, similar to Connor and Korajczyk (1993), Lee and Robinson (2016) and Ma et al. (2021), one can assume that there exists a reordering of the cross-sectional units such that their dependence can be characterized by the uniform mixing condition mentioned above, and the conclusion of Proposition 1 will still hold.

To establish the convergence rates of the estimated factors and loading functions, we need to impose more assumptions.

Assumption 2. *Let M be a generic bounded constant.*
(i) Let $\boldsymbol{\Sigma}_\phi = \mathbb{E}[\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)']$. Then there exist constants c_1, c_2 such that $0 < c_1 \leq \lambda_{\min}(\boldsymbol{\Sigma}_\phi) \leq \lambda_{\max}(\boldsymbol{\Sigma}_\phi) \leq c_2 < \infty$ for all n.
(ii) $k_n^2/n \to 0$ as $n \to \infty$.
(iii) There exist $c > 0$ such that $\lambda_{\min}(\boldsymbol{F}'\boldsymbol{F}/T) > c$ for all T.
(iv) $\hat{\boldsymbol{\Sigma}}_g \equiv n^{-1}\sum_{i=1}^n \boldsymbol{g}(\boldsymbol{x}_i)\boldsymbol{g}(\boldsymbol{x}_i)' \xrightarrow{P} \boldsymbol{\Sigma}_g > 0$ as $n \to \infty$.
(v) The eigenvalues of $\boldsymbol{\Sigma}_g \cdot \boldsymbol{F}'\boldsymbol{F}/T$ are distinct.

The conditions in Assumption 2 are all standard in literature of factor models and sieve estimation. In particular, Assumption 2(ii) strengthens Assumption 1(v), and Assumption 2(iii) implicitly requires that $T \geq R$. In comparison, Assumption A0 of Ma et al. (2021) imposes that $\liminf_{T\to\infty} |T^{-1}\sum_{t=1}^T f_{tr}| > 0$ for all $r = 1, \ldots, R$, which excludes the possibility that the underlying time series that generates $\boldsymbol{F}$ has mean 0. The following theorem gives the rates of convergence of the estimated factors and the estimated loading functions.

Theorem 1. *Let $\hat{\boldsymbol{\Omega}}$ be the diagonal matrix whose elements are the eigenvalues of $\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}}/(nT)$, and define $\hat{\boldsymbol{H}} = \hat{\boldsymbol{\Sigma}}_g(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)\hat{\boldsymbol{\Omega}}^{-1}$. Then under Assumptions 1 and 2, the following results hold either T is fixed or $T \to \infty$ as $n \to \infty$,:*
(i) $\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T} = O_P(\varepsilon_{nT})$.
(ii) $\|\hat{\boldsymbol{G}}(\boldsymbol{X}) - \boldsymbol{G}(\boldsymbol{X})(\hat{\boldsymbol{H}}')^{-1}\|/\sqrt{n} = O_P(\varepsilon_{nT})$.
(iii) $\sup_{\boldsymbol{x}\in\mathcal{X}} \|\hat{\boldsymbol{g}}(\boldsymbol{x}) - \hat{\boldsymbol{H}}^{-1}\boldsymbol{g}(\boldsymbol{x})\| = O_P(\sqrt{k_n}\varepsilon_{nT})$.

First, it is worth noting that Theorem 1 (and Theorem 2 below) is obtained without any restrictions on the time series dependence of the idiosyncratic errors, while both Fan et al. (2016) and Ma et al. (2021) imposed some kind of weak-time-series-dependence conditions. Second, no moment restrictions are imposed on u_{it} in our assumptions, while Assumption 3.4 of Fan et al. (2016) requires the error terms to have exponential tails. Third, the convergence rates given in Theorem 1 are generally slower than those of Fan et al. (2016), mainly because of the difference between the sieve quantile estimators and the sieve least square estimators. In fact, in the proof of Theorem 1, we only used the uniform convergence rate of $\hat{\boldsymbol{a}}_t$ (see Lemma 1 in the appendix). By exploring the Bahadur representation of $\hat{\boldsymbol{a}}_t$, the convergence rate of the estimated loading functions can be improved (see Theorem 3 below) when T is large, and the convergence rate of the estimated factors can be greatly improved even when T is fixed if we impose the following assumptions.

Assumption 3. *Let L be a generic bounded constant and let $\mathsf{f}(\cdot)$ denote the cumulative distribution function of u_{it}.*
(i) For each i, $\boldsymbol{x}_i$ is independent of $(u_{i1}, \ldots, u_{iT})$.

(ii) $|\mathsf{f}(c) - \mathsf{f}(0)| \leq L|c|$ *for any* c *in a neighborhood of* 0.
(iii) Equation (4) *holds for some* $\alpha \geq 3$.

Assumption 3(i) essentially requires that the observed characteristics only affect the location but not the scale of the distributions of y_{it}. In this case, the leading term in the Bahadur representation of $\hat{\boldsymbol{a}}_t$ has a structural that is similar to the least square estimators — see Lemma 2 in the appendix. This allows us to obtain an improved convergence rate of $\hat{\boldsymbol{F}}$ that is as fast as the PPCA estimators (see Theorem 4.1 of Fan et al. (2016)).

Theorem 2. *Let* $\eta_{nT} = \sqrt{\ln(k_n^{-1/4}\varepsilon_{nT}^{-1/2})} \cdot k_n^{5/4}\varepsilon_{nT}^{1/2}n^{-1/2}$. *Under Assumptions 1 to 3, we have*

$$\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T} = O_P\left(n^{-1/2} \vee k_n^{-\alpha} \vee \eta_{nT} \vee \varepsilon_{nT}^2\right).$$

Moreover, if $T \asymp n^{\gamma_1}$ *and* $k_n \asymp n^{1/(6+\gamma_2)}$ *for some* $\gamma_1 \geq 0$ *and* $\gamma_2 > 0$, *then*

$$\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T} = O_P\left(n^{-1/2} \vee k_n^{-\alpha}\right).$$

Remark 4. *The term* η_{nT} *in Theorem 2 represents the higher order terms in the Bahadur representation of* $\hat{\boldsymbol{a}}_t$. *When* α *is large,* η_{nT} *is approximately equal to* $k_n^{3/2}n^{-3/4}$. *This slightly unusual expression of* η_{nT} *is mainly due to the non-smoothness of the check function. Similar terms can be found in Theorem 2 of Horowitz and Lee (2005), Theorem 3.2 of Kato et al. (2012) and Theorem 2 of Ma et al. (2021).*

3.2 Asymptotic Distribution

Define $\boldsymbol{\Sigma}_{\mathsf{f}\phi} = \mathbb{E}[\mathsf{f}(0|\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)']$ and $\sigma_{k_n}^2 = \boldsymbol{\phi}_{k_n}'(\boldsymbol{x})\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\Sigma}_{\phi}\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\phi}_{k_n}(\boldsymbol{x})$.

Assumption 4. *Let* L *be a generic bounded constant.*
(i) $u_{i1}, \ldots, u_{iT}$ *are independent conditional on* $\boldsymbol{x}_i$.
(ii) $|\mathsf{f}(c|\boldsymbol{x}) - \mathsf{f}(0|\boldsymbol{x})| \leq L|c|$ *for any* c *in a neighborhood of* 0 *and any* $\boldsymbol{x} \in \mathcal{X}$.
(iii) There exist constants c_1, c_2 *such that* $0 < c_1 \leq \lambda_{\min}(\boldsymbol{\Sigma}_{\mathsf{f}\phi}) \leq \lambda_{\max}(\boldsymbol{\Sigma}_{\mathsf{f}\phi}) \leq c_2 < \infty$ *for all* k_n.
(iv) $(nT)^{1/2}k_n^{1/2-\alpha}\sigma_{k_n}^{-1} = o(1)$ *and* $(nT)^{1/2}k_n^{1/2}\eta_{nT}\sigma_{k_n}^{-1} = o(1)$.

Assumption 4(i) is imposed for simplicity, and it can be replaced by $\beta-$mixing conditions at the cost of more complex asymptotic covariance matrices. When $\sigma_{k_n} \asymp k_n^{1/2}$ and T is fixed, Assumption 4(iv) essentially requires that $n^{1/2}k_n^{-\alpha} = o(1)$ and $n^{1/2}\eta_{nT} = o(1)$, or $k_n^6 \ll n \ll k_n^{2\alpha}$. As a result, we need (4) to hold with $\alpha > 3$. The other conditions of Assumption 4 are standard — see Assumptions 3 and 5 of Horowitz and Lee (2005) for example. Then we can establish the asymptotic distribution of the estimated loading functions.

Theorem 3. *If Assumptions 1, 2 and 4 hold, then for any $\boldsymbol{x} \in \mathcal{X}$, we have*

$$\boldsymbol{\Sigma}_{T,\tau}^{-1/2}(\hat{\boldsymbol{H}}')^{-1} \cdot \frac{\sqrt{nT}}{\sigma_{k_n}} \left(\hat{\boldsymbol{g}}(\boldsymbol{x}) - (\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'\boldsymbol{g}(\boldsymbol{x})\right) \xrightarrow{d} N(0, \boldsymbol{I}_R),$$

where $\boldsymbol{\Sigma}_{T,\tau} = \tau(1-\tau)(\boldsymbol{F}'\boldsymbol{F}/T)$.

The asymptotic distribution of $\hat{\boldsymbol{F}}$ is more difficult to derive, especially in the case where n and T go to infinity simultaneously. Instead, we consider the following updated estimator for the factors:

$$\tilde{\boldsymbol{F}} = \hat{\boldsymbol{Y}}'\hat{\boldsymbol{G}}(\boldsymbol{X}) \cdot (\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X}))^{-1}.$$

Moreover, let $\tilde{\boldsymbol{H}} = (\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}$ and

$$\boldsymbol{\Xi}_\tau = \tau(1-\tau) \cdot \boldsymbol{\Sigma}_g^{-1}\mathbb{E}[\boldsymbol{g}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)']\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\Sigma}_\phi\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\mathbb{E}[\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\boldsymbol{g}(\boldsymbol{x}_i)']\boldsymbol{\Sigma}_g^{-1}.$$

The following assumption is needed to derive the asymptotic distribution of $\tilde{\boldsymbol{f}}_t$.

Assumption 5. *Conditions (i) to (iii) of Assumption 4 hold and $n^{1/2}k_n^{-\alpha}/\|\boldsymbol{\Xi}_\tau\|^{1/2} = o(1)$, $n^{1/2}\eta_{nT}/\|\boldsymbol{\Xi}_\tau\|^{1/2} = o(1)$, $\varepsilon_{nT}\sqrt{k_n} = o(1)$.*

Theorem 4. *If Assumptions 1, 2 and 5 hold, then for all $t = 1, \dots, T$,*

$$\boldsymbol{\Xi}_\tau^{-1/2}(\hat{\boldsymbol{H}}')^{-1}\sqrt{n}(\tilde{\boldsymbol{f}}_t - \tilde{\boldsymbol{H}}'\boldsymbol{f}_t) \xrightarrow{d} N(0, \boldsymbol{I}_R).$$

When $\|\boldsymbol{\Xi}_\tau\| \asymp k_n$, the convergence rate of $\tilde{\boldsymbol{f}}_t$ is $O_P(\sqrt{n/k_n})$, and Assumption 5 requires that $n^{1/2}k_n^{-\alpha-1/2} = o(1)$ and $n^{1/2}\eta_{nT}k_n^{-1/2} = o(1)$, or $k_n^4 \ll n \ll k_n^{2\alpha+1}$. As a result, we need (4) to hold with $\alpha \geq 2$. Alternatively, if Assumption 3(i) holds, it can be shown that

$$\|\boldsymbol{\Xi}_\tau - \tau(1-\tau) \cdot \boldsymbol{\Sigma}_g^{-1} \cdot \mathsf{f}^{-2}(0)\| = O(k_n^{-\alpha}).$$

In this case, the convergence rate of $\tilde{\boldsymbol{f}}_t$ is $\sqrt{n}$ for each t.

Remark 5. *Similar to Proposition 1 of Bai (2003), it can be shown that $\hat{\boldsymbol{H}}$, $\tilde{\boldsymbol{H}}$ and $\boldsymbol{F}'\hat{\boldsymbol{F}}/T$ all converge in probability to some positive definite matrices as $n, T \to \infty$. In particular, if $\boldsymbol{F}'\boldsymbol{F}/T = \boldsymbol{I}_R$ and $\hat{\boldsymbol{\Sigma}}_g$ is diagonal, the probability limits of $\hat{\boldsymbol{H}}$, $\tilde{\boldsymbol{H}}$ and $\boldsymbol{F}'\hat{\boldsymbol{F}}/T$ are all equal to $\boldsymbol{I}_R$.*

Remark 6. *Note that both Theorem 3 and Theorem 4 hold either T is fixed or $T \to \infty$ as $n \to \infty$. In the latter case, Chen et al. (2021) showed that if $n \asymp T$, their estimators of the quantile factors are $\sqrt{n}$-consistent and asymptotically normal under more general conditions. Thus, if T is as large as n and the main objects of interests are the quantile factors, the estimators of Chen* *et al. (2021) seems to be the better choice. However, when T is small, Theorem 4 shows that the estimators proposed in this paper remain consistent and asymptotically normal as long as n is large.*

3.3 Estimating the Number of Factors

Intuitively, since $\hat{\boldsymbol{Y}} = \boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}} \approx \boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{A}_0 = \boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{B}_0\boldsymbol{F}' \approx \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'$, the rank of $\hat{\boldsymbol{Y}}$ is asymptotically equal to R. Let $\hat{\rho}_1, \ldots, \hat{\rho}_{\bar{R}}$ be the $\bar{R}$ largest eigenvalues of $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Y}}'/(nT)$ in descending order. Then the estimator of R is given by the number of non-vanishing eigenvalues of $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Y}}'/(nT)$, i.e.,

$$\hat{R} = \sum_{j=1}^{\bar{R}} \mathbf{1}\{\hat{\rho}_j > p_n\}, \tag{8}$$

where $\{p_n\}$ is a sequence of non-increasing positive constants. The following theorem provides conditions on the threshold p_n to establish the consistency of $\hat{R}$.

Theorem 5. *Suppose that $\bar{R} \geq R$ and $p_n \to 0$, $p_n \varepsilon_{nT}^{-1} \to \infty$ as $n \to \infty$, then under Assumptions 1 and 2, we have $P[\hat{R} = R] \to 1$ as $n \to \infty$.*

To prove Theorem 5, we show that the largest R eigenvalues of $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Y}}'/(nT)$ converge in probability to some positive constants, while the remaining eigenvalues are all $O_P(\varepsilon_{nT})$. Then the decreasing sequence $\{p_n\}$ is chosen to dominate the vanishing eigenvalues in the limit. Again, this result also holds even when T is fixed.

In theory, the choice of p_n is determined by α, which depends on the smoothness of the unknown quantile loading functions. Thus, a conservative choice of p_n can be obtained by assuming that $\alpha = 1$. In this case, $\varepsilon_{nT} = (k_n^{1/2} n^{-1/2} \vee k_n^{-1}) \ln T$, and the optimal choice of k_n is $k_n^* \asymp n^{1/3}$. Thus, to satisfy the condition of Theorem 5, we need $p_n \gg n^{-1/3} \ln T$. The following choice is recommended in practice:

$$p_n = d \cdot \hat{\rho}_1^{1/2} \cdot n^{-1/4} \ln T, \tag{9}$$

where d is a positive constant and $\hat{\rho}_1^{1/2}$ plays the role of a normalization factor.

Remark 7. *Alternatively, to avoid the choice of the threshold sequence $\{p_n\}$, one can use the idea of Ahn and Horenstein (2013) to estimate the number of factor by maximizing the ratios of consecutive eigenvalues, i.e.,*

$$\tilde{R} = \operatorname*{arg\,max}_{j=1,\ldots,\bar{R}} \frac{\hat{\rho}_j}{\hat{\rho}_{j+1}}.$$

This is the estimator considered by Fan et al. (2016) in the context of approximate factor models. In particular, Fan et al. (2016) required the error terms to be sub-Gaussian. However, to formally prove the consistency of this estimator in the context of QFM is technically challenging.

4 Simulations

4.1 Estimating The Number of Factors

Consider the following DGP:

$$y_{it} = \sum_{r=1}^{3} \lambda_{ir} f_{tr} + \left(x_{i1}^2 + x_{i2}^2 + x_{i3}^2\right) u_{it},$$

where $f_{t1} = 1$, $f_{t2}, f_{t3} \sim i.i.d\ N(0,1)$. Suppose that the number of characteristics is 5 and all characteristics x_{id} are drawn independently from the uniform distribution: $U[-1,1]$. Let $g_1(z) = sin(2\pi z)$, $g_2(z) = sin(\pi z)$ and $g_3(z) = cos(\pi z)$. Further, assume that

$$\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id}), \quad \lambda_{i2} = \sum_{d=1,2} g_2(x_{id}), \quad \lambda_{i3} = \sum_{d=3,4} g_3(x_{id}).$$

For u_{it}, they are i.i.d draws from three different distributions: standard normal distribution: $N(0,1)$, Student's t distribution with 3 degrees of freedom: $T(3)$, and standard Cauchy distribution: Cauchy(0,1). In the first step quantile sieve estimation, we set $k_n = n^{1/3}$ and use the *Chebyshev polynomials of the second kind* as the basis functions. Moreover, to use the proposed estimator for the number of factors in (8), the threshold p_n is chosen as in (9) with $d = 1/4$.

Table 1 presents the estimation results for $\tau \in \{0.25, 0.5, 0.75\}$, $T \in \{5, 10\}$ and $n \in \{50, 100, 200, 1000\}$ from 1000 replications. For each combination of τ, n and T, the reported results are [frequency of $\hat{R} < R$; frequency of $\hat{R} = R$; frequency of $\hat{R} > R$]. For comparison purpose, Table 2 reports the results where the numbers of factors are estimated using the eigen-ratio estimator mentioned in Remark 7.

There are three main takeaways from the results. First, both methods can accurately estimate the number of factors when T is small and N is large, which confirms our claim that our estimator is consistent even when T is fixed. Second, when N is large (=1000), both estimators perform well even when the errors follow the standard Cauchy distribution, providing support for the claim that our estimator is consistent without moment restrictions on the errors. Third, in general, the two estimators are close when $N = 1000$, but our estimator performs much better than the eigen-ratio estimator when N is not large.

4.2 Estimating The Factors

4.2.1 Comparison with PCA, PPCA and QFA

Following Chen et al. (2021), we consider the following DGP:

$$y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it},$$

where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. Suppose that the number of characteristics is 5 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2, 3, 4, 5$) are independently drawn from the uniform distribution: $U[-1, 1]$. Let $g_1(x) = sin(2\pi x),\ g_2(x) = sin(\pi x)\ and\ g_3(x) = |cos(\pi x)|$. The factor loading functions are generated as $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id}),\ \lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ and $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. For u_{it}, again we consider three different distributions as in Section 4.1. Note that in this DGP, there are two mean factors f_{t1} and f_{t2} that affect the mean of Y_{it} and one scale factor f_{t3} that affects the variance of Y_{it}.

First, we focus on the estimation of the two mean factors: f_{t1} and f_{t2}. Four competing methods are considered: (i) the proposed method in this paper with $\tau = 0.5$ (labelled as QPCA); (ii) The quantile factor analysis estimator of Chen et al. (2021) with $\tau = 0.5$ (labelled as QFA); (iii) the PPCA estimator proposed by Fan et al. (2016); (iv) the PCA estimator of Bai and Ng (2002). For the first two methods, the choice of k_n and the basis functions are the same as in Section 4.1.

Regarding the choices of N and T, two scenarios are considered:

(i) Fix $T = 10, 50$ and let n increase from 50 to 500.

(ii) Fix $n = 100, 200$ and let T increase from 5 to 200.

For each estimation method, the number of factors ($R = 2$ at $\tau = 0.5$) is assumed to be known, and we report the average Frobenius error: $\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T}$ from 1000 replications, where $\hat{\boldsymbol{H}}$ is the associated rotation matrix for each estimator.

The results for fixed T and increasing n are reported in Figure 1. It can be seen that when T is small ($T = 10$), the PCA and QFA estimators perform worse than the PPCA and QPCA estimators when u_{it} is drawn from the standard normal or $T(3)$ distribution. In particular, when u_{it} follows the standard Cauchy distribution, the QPCA estimator performs much better than the other three methods. These findings are in line with our theoretical results that the QPCA estimator is consistent even when T is fixed or the moments of u_{it} does not exist.

When T is relatively large ($T = 50$) and the distribution of u_{it} has a thin tail (Normal distribution), all the estimators considered here behave similarly as long as $N \geq 100$. However, when u_{it} follows the $T(3)$ distribution, the PCA estimator suffers a much larger estimation error compared with the other methods. In the extreme case where u_{it} follows the standard Cauchy distribution, the two methods based on quantile regressions are the obvious winners, and it should be noted that the performances of the QFA and QPCA estimators are very close as long as $N \geq 200$.

The results for fixed n and increasing T are reported in Figure 2. Besides those findings from Figure 1, the main takeaway from Figure 2 is that the QPCA estimator is the most robust one against heavy-tailed distributions when T is small, but the QFA estimator performs slightly better as T increases.

Second, we estimate all the three factors and focus on the scale factor f_{t3}. Now consider the QFA and QPCA estimators with $\tau = 0.25, 0.75$, and the PCA and PPCA estimators are also included for comparison. To save space, we consider $T \in \{10, 50\}$ and $n \in \{50, 100, 200\}$. For each case, three factors are estimated, which are denoted as $\hat{F}^{\tau}_{QPCA}, \hat{F}^{\tau}_{QFA}, \hat{F}_{PPCA}, \hat{F}_{PCA}$. Then each of the true factors is regressed on these estimated factors and the adjusted R^2s are calculated. The whole procedure is repeated 1000 times and the averages of the adjusted R^2s are reported in Table 3 to Table 5.

The results for the QPCA estimator are shown in Table 3. It can be observed that the QPCA estimator perform very well in estimating all the three factors. It should be noted that the estimation of the scale factor f_{t3} is not as good as the mean factors f_{t1}, f_{t2} when n is small but such differences vanish as n increases. The results for the QFA estimator are shown in Table 4 and the results for the PCA and PPCA estimators are presented in Table 5. It can be found that the QFA estimator performs poorly in estimating the scale factor f_{t3} when T is small ($T = 10$) while it is similar to the QPCA estimator when T is relatively large ($T = 50$). Moreover, from Table 5 it can be seen that the PCA and PPCA estimators fail to capture the scale factor f_{t3} in any case.

4.2.2 Comparison with Ma et al. (2021)

In the previous comparison we didn't consider the estimator of Ma et al. (2021), because the performance of their estimator is close to our QPCA estimator when the number of characteristics is larger than or equal to the number of factors. In this part, we focus on the differences between the QPCA estimator and the estimator of Ma et al. (2021) (labelled as SQFA) when the number of characteristics is less than the number factors ($D < R$).

Consider the following DGP:

$$y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it},$$

where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\, N(0, 1)$. Suppose that the number of characteristics is 2 and all characteristics $x_{id} (i = 1, ...N$ and $d = 1, 2)$ are independently drawn from uniform distribution:

$U[-1,1]$. Let $g_1(x) = sin(2\pi x)$, $g_2(x) = sin(\pi x)$ *and* $g_3(x) = |cos(\pi x)|$. Moreover, let $\lambda_{i1} = \sum_{d=1,2} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ *and* $\lambda_{i3} = \sum_{d=1,2} g_3(x_{id})$. Again, u_{it} are generated from three different distributions as in Section 4.1.

For each case we consider $\tau \in \{0.25, 0.5, 075\}$, $T \in \{10, 50\}$, $N \in \{50, 100, 200, 500\}$. Note that when $\tau = 0.5$ there are only two mean factors because f_{t3} does not affect the median of y_{it} while when $\tau = 0.25, 0.75$ there are two mean factors and one scale factor. Moreover, we assume that R is known. For each τ, R factors are estimated using QPCA. Note that the SQFA method chooses the number of factors as the number of characteristics by default, so only two factors are estimated. Moreover, the choices of the basis functions and k_n are the same as in Section 4.1.

Each of the true factors is regressed on these estimated factors and the adjusted R^2s are calculated. The whole procedure is repeated 1000 times and we report the averages of adjusted R^2s. The results are shown in Table 6 for the QPCA estimator and in Table 7 for the SQFA estimator. When it comes to the estimation of the volatility factor, it is not surprising to see that the QPCA estimator outperforms the SQFA estimator which only estimates $D = 2$ factors. The main finding here is that the QPCA estimator also performs much better than the SQFA estimator in estimating the mean factors in almost all cases.

4.3 Estimating The Loading Functions

In this section, we focus on the estimation of the quantile loading functions. Consider the following DGP:

$$y_{it} = \lambda_{i1} f_{t1} + (\lambda_{i2} f_{t2}) u_{it},$$

where $f_{t2} = |g_t|$ and $f_{t1}, g_t \sim i.i.d\ N(0, 1)$. The number of characteristics is 2 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2$) are independently drawn from uniform distribution: $U[-1, 1]$. Let $g_{11}(x) = sin(2\pi x), g_{21}(x) = 0, g_{12}(x) = sin(\pi x)$, $g_{22}(x) = cos^2(\pi x)$ and $\lambda_{i1} = g_{11}(x_{i1}) + g_{12}(x_{i2})$, $\lambda_{i2} = g_{21}(x_{i1}) + g_{22}(x_{i2})$. Again, u_{it} are drawn independently from the standard normal distribution or Student's t distribution with 3 degrees of freedom. Note that in this DGP, there is one mean factor f_{1t} and a scale factor f_{2t}. Moreover, g_{11} and g_{12} are associated with the mean factor and they are quantile-invariant, while g_{21} and g_{22} are associated with the scale factor and they are quantile-dependent. In particular, suppose the lower quantile of u_{it} is Q_τ, then the true loading function $g_{\tau,2d}(x) = \mathsf{Q}_\tau \cdot g_{2d}(x)$.

Here we only consider our QPCA estimator and set $n = 500, T = 10$. The choices of basis functions and k_n are the same as in Section 4.1. For each distribution of the error terms, we estimate the loading functions at $\tau = 0.25, 0.5, 0.75$. We take 201 points on [-1,1] with equal distance and calculate the estimated function value at these points. This procedure is repeated 1000 times and we report the lower 5% and 95% quantiles of 1000 replications at each point.

Figures 3 and 4 show the results when the errors follows the standard normal distribution, and Figures 5 and 6 show the results when errors are drawn from $T(3)$. From these figures, it can be seen that the QPCA estimator performs very well in estimating the true loading functions even when T is not large.

5 Application

In this section the proposed estimation method is applied to study the factor structure of security returns. Following Fan et al. (2016), the dataset we use consists of the daily returns of S&P500 index securities that have complete daily closing price records from 2005 to 2013.[2] There are 355 stocks included in this dataset. The book value and market capitalization of each stock are also obtained from Compustat. Moreover, the 1-month US treasury bond rate is used as the risk-free rate to calculate the daily excess return of each stock.

We consider four characteristics as in Connor et al. (2012), Fan et al. (2016), and Ma et al. (2021): *size, value, momentum* and *volatility.* All characteristics are standardized so their means are 0 and their standard deviations are equal to 1. Similar to Fan et al. (2016), we analyze the data in the first quarter of 2006, which includes $T = 62$ observations. The second Chebyshev polynomials are used as the basis functions in the sieve regressions and $k_n = 4$.

First, Table 8 shows the estimated number of mean factors using the eigen-ratio estimator proposed by Fan et al. (2016) and the estimated numbers of quantile factors using the proposed estimator in this paper for $\tau \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$. The five largest eigenvalues of $\hat{Y}\hat{Y}'$ and the threshold p_n are also presented in this table. In addition, the estimated numbers of quantile factors using the rank-minimization estimator proposed by Chen et al. (2021) are also reported in the last column. The results provide strong evidence that there is only 1 mean factor and 1 quantile factor at each quantile.

Second, Table 9 shows the the correlation coefficients between the estimated mean factor by PPCA and the estimated quantile factors by QPCA for $\tau \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$. The sample means of each estimated factor are also reported in the last column. Figure 7 provides plots these factors. It can be observed that these factors have high correlations while their means are different.

Next, Figure 8 shows the estimated loading functions of the four characteristics using PPCA and QPCA at $\tau = 0.5$. It can be observed that the estimated loading functions by these two method are quite similar, indicating that the idiosyncratic errors of the stock returns have symmetric distributions.

Finally, Figure 9 plots the estimated loading functions of the four characteristics using QPCA

[2] This dataset is downloaded from CRSP (Center for Research in Security Prices).

at different quantiles. In general, these functions feature considerable variations across the values of the characteristics and also across quantiles. Three main results emerge. First, the loading functions of size and volatility at all quantiles seem to be monotone, while for value and momentum, their loading functions have a clear non-linear pattern. Second, for all characteristics, their quantile loading functions at $\tau = 0.25$ and $\tau = 0.5$ are very close. Third, it can be seen that in general the loading functions at the tails ($\tau = 0.05, 0.95$) have greater curvatures than the loading functions at the other quantiles.

6 Conclusions

This paper proposes a three-stage estimation method for characteristic-based quantile factor models. The convergence rates of the proposed estimators are established, and the asymptotic distributions of the estimated factors and loading functions are derived under very general conditions. Compared with the existing estimation methods, the proposed estimators in this paper are easy to implement in practice, consistent for fixed T as long as N goes to infinity, and robust to heavy tails and outliers of the idiosyncratic errors. Moreover, the number of quantile factors are allowed to be different from the number of the characteristic, and it can be consistently estimated using a new estimator proposed in this paper.

Simulation results show that the proposed estimators perform very well in finite samples, especially when the number of cross section observations is large. An application of the estimators to a dataset consisting of individual stock returns reveals that the quantile factor loadings are nonlinear functions of some observed characteristics, and these functions have considerable variations across quantiles.

For the tractability of the problem, this paper assumes that the quantile factor loadings can be fully explained by the observed characteristics. Admittedly, this is a restrictive assumption. Relaxing this assumption and allowing the factor loadings to be functions of other unobserved characteristics is a challenging task in the context of quantile regressions. This interesting question is left for future research.

A Proofs of the Main Results

Proof of Proposition 1:

Proof. For any $\theta \in \Theta$, define $K(\theta, \theta_{0t}) = \mathbb{E}(L_n(\theta)) = \mathbb{E}[l(\theta, y_{it}, \boldsymbol{x}_i)]$. Under Assumption 1(iv), it can be shown that $K(\theta, \theta_{0t}) \asymp d(\theta, \theta_{0t})^2$. For the finite dimensional linear sieve spaces Θ_n, it can be shown that Condition A.3 of Chen and Shen (1998) is satisfied with $\delta_n = \sqrt{k_n/n}$ (see Section 3.3 of Chen (2007)). By the definition of d and the property of check function[3], it is easy to see that

$$\begin{aligned}\sup_{\theta\in\Theta_n, d(\theta,\theta_{0t})\leq\varepsilon} \mathrm{Var}\left[l(\theta, y_{it}, \boldsymbol{x}_i)\right] &\leq \sup_{\theta\in\Theta_n, d(\theta,\theta_{0t})\leq\varepsilon} \mathbb{E}\left[l(\theta, y_{it}, \boldsymbol{x}_i)\right]^2 \\ &\lesssim \sup_{\theta\in\Theta_n, d(\theta,\theta_{0t})\leq\varepsilon} \mathbb{E}\left(\theta(\boldsymbol{x}_i) - \theta_{0t}(\boldsymbol{x}_i)\right)^2 \leq \varepsilon^2.\end{aligned}$$

Thus, Condition A.2 of Chen and Shen (1998) is also satisfied. By Assumption 1(iii) we have $\sup_{\theta\in\Theta} |l(\theta, y_{it}, \boldsymbol{x}_i)| \lesssim \sup_{\theta\in\Theta} \sup_{\mathcal{X}} |\theta(\boldsymbol{x}) - \theta_{0t}(\boldsymbol{x})| < \infty$. Assumption 1(ii) implies that $d(\pi_n\theta_{0t}, \theta_{0t}) = \sqrt{\mathbb{E}\left(\pi_n\theta_{0t}(\boldsymbol{x}_i) - \theta_{0t}(\boldsymbol{x}_i)\right)^2} = O(k_n^{-\alpha})$. Therefore, it follows from Corollary 1 of Chen and Shen (1998) that

$$P\left[\max_t d(\hat{\theta}_{nt}, \theta_{0t}) \geq C\varepsilon_{nT}\right] \leq \sum_{t=1}^{T} P\left[d(\hat{\theta}_{nt}, \theta_{0t}) \geq C\varepsilon_{nT}\right] \leq c_1 \exp\left\{C^2 \ln T(1 - c_2 n\varepsilon_n^2)\right\}$$

for any $C \geq 1$. Therefore, the desired result follows from the above inequality since $n\varepsilon_n^2 \geq k_n$. $\square$

Lemma 1. *If Assumption 1 and Assumption 2(i) hold, and ε_n is defined as in Assumption 1, then:*
(i) $\max_{1\leq t\leq T} \|\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t}\| = O_P(\varepsilon_{nT})$;
(ii) Let $\hat{\boldsymbol{V}} \equiv \hat{\boldsymbol{Y}} - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'$, *then* $(nT)^{-1/2}\|\hat{\boldsymbol{V}}\| = O_P(\varepsilon_{nT})$.

Proof. By Assumption 1 and Assumption 2(i),

$$\begin{aligned}d(\hat{\theta}_{nt}, \theta_{0t})^2 = \int_{\mathcal{X}} \left(\hat{\theta}_{nt}(\boldsymbol{x}) - \theta_{0t}(\boldsymbol{x})\right)^2 d\mathsf{F}_x(\boldsymbol{x}) &= \int_{\mathcal{X}} \left(\hat{\theta}_{nt}(\boldsymbol{x}) - \pi_n\theta_{0t}(\boldsymbol{x})\right)^2 d\mathsf{F}_x(\boldsymbol{x}) + O_P(\varepsilon_{nT}k_n^{-\alpha}) \\ &= (\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t})'\boldsymbol{\Sigma}_\phi(\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t}) + O_P(\varepsilon_{nT}k_n^{-\alpha}) \geq c_1\|\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t}\|^2 + O_P(\varepsilon_{nT}k_n^{-\alpha})\end{aligned}$$

where $c_1 > 0$, and the $O_P(\varepsilon_{nT}k_n^{-\alpha})$ in the above equation is uniform in t. It then follows from Proposition 1 that $\max_{1\leq t\leq T} \|\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t}\|^2 = O_P(\varepsilon_{nT}^2)$.

[3]Note that $|\rho_\tau(u_1) - \rho_\tau(u_2)| \leq 2|u_1 - u_2|$.

Next, note that

$$
\begin{aligned}
(nT)^{-1}\|\hat{\boldsymbol{V}}\|^2 &\leq \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\hat{\theta}_{nt}(\boldsymbol{x}_i)-\pi_n\theta_{0t}(\boldsymbol{x}_i)\right)^2+O_P(k_n^{-2\alpha})\\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left((\hat{\boldsymbol{a}}_t-\boldsymbol{a}_{0t})'\,\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\right)^2+O_P(k_n^{-2\alpha})\\
&\leq T^{-1}\sum_{t=1}^{T}\|\hat{\boldsymbol{a}}_t-\boldsymbol{a}_{0t}\|^2\cdot\lambda_{\max}\left(\hat{\boldsymbol{\Sigma}}_\phi\right)+O_P(k_n^{-2\alpha})\\
&\leq \max_{1\leq t\leq T}\|\hat{\boldsymbol{a}}_t-\boldsymbol{a}_{0t}\|^2\cdot\lambda_{\max}\left(\hat{\boldsymbol{\Sigma}}_\phi\right)+O_P(k_n^{-2\alpha})
\end{aligned}
$$

where $\hat{\boldsymbol{\Sigma}}_\phi \equiv n^{-1}\sum_{i=1}^{n}\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)'$. Since Assumption 1(iii) implies that $\sup_{\mathcal{X}}\|\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\| = \sqrt{k_n}$, similar to the proof of Theorem 1 in Newey (1997), one can show that $\|\hat{\boldsymbol{\Sigma}}_\phi-\boldsymbol{\Sigma}_\phi\| = o_P(1)$ under Assumption 2, and therefore we have $\lambda_{\max}(\hat{\boldsymbol{\Sigma}}_\phi) = O_P(1)$. This completes the proof. □

Proof of Theorem 1:

Proof. Write $\hat{\boldsymbol{Y}} = \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}' + \hat{\boldsymbol{V}}$ where $\hat{\boldsymbol{V}}$ is as defined in Lemma 1. Let $\boldsymbol{\Omega}_R$ be the diagonal matrix whose elements are the eigenvalues of $\boldsymbol{\Sigma}_g\cdot\boldsymbol{F}'\boldsymbol{F}/T$. Note that

$$
\begin{aligned}
\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}}/(nT) = \boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'/(nT) + \hat{\boldsymbol{V}}'\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'/(nT)&\\
+\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{V}}/(nT)+\hat{\boldsymbol{V}}'\hat{\boldsymbol{V}}/(nT).&\quad \text{(A.1)}
\end{aligned}
$$

It then follows from Assumption 2(iv), Assumption 1(i) and Lemma 1 that:

$$
\begin{aligned}
&\|\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}}/(nT)-\boldsymbol{F}\boldsymbol{\Sigma}_g\boldsymbol{F}'/T\|\\
\leq\ & o_P(1)+2\|\hat{\boldsymbol{V}}\|/\sqrt{nT}\cdot\|\boldsymbol{G}(\boldsymbol{X})\|/\sqrt{n}\cdot\|\boldsymbol{F}\|/\sqrt{T}+\|\hat{\boldsymbol{V}}\|^2/(nT)\\
=\ & o_P(1)+O_P(\varepsilon_n).
\end{aligned}
$$

By the Wielandt-Hoffman inequality, we have $\|\hat{\boldsymbol{\Omega}}-\boldsymbol{\Omega}\| = o_P(1)$. It then follows from Assumption 2(iii) and 2(iv) that $\lambda_{\min}(\hat{\boldsymbol{\Omega}}) > 0$ with probability approaching 1.

By the definition of $\hat{\boldsymbol{F}}$, $\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}}/(nT)\hat{\boldsymbol{F}} = \hat{\boldsymbol{F}}\hat{\boldsymbol{\Omega}}$, it then follows from (A.1) that

$$
\hat{\boldsymbol{F}} = \boldsymbol{F}\hat{\boldsymbol{H}} + \hat{\boldsymbol{V}}'\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\hat{\boldsymbol{F}}/(nT)\hat{\boldsymbol{\Omega}}^{-1} + \boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{V}}\hat{\boldsymbol{F}}/(nT)\hat{\boldsymbol{\Omega}}^{-1} + \hat{\boldsymbol{V}}'\hat{\boldsymbol{V}}/(nT)\hat{\boldsymbol{F}}\hat{\boldsymbol{\Omega}}^{-1}. \quad \text{(A.2)}
$$

Thus, it follows from (A.2) and Lemma 1 that

$$
\|\hat{\boldsymbol{F}}-\boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T} \leq 2O_P(1)\cdot\frac{\|\hat{\boldsymbol{V}}\|}{\sqrt{nT}}\cdot\frac{\|\boldsymbol{F}\|}{\sqrt{T}}\cdot\frac{\|\hat{\boldsymbol{F}}\|}{\sqrt{T}}\cdot\frac{\|\boldsymbol{G}(\boldsymbol{X})\|}{\sqrt{n}} + O_P(1)\cdot\frac{\|\hat{\boldsymbol{F}}\|}{\sqrt{T}}\cdot\frac{\|\hat{\boldsymbol{V}}\|^2}{nT} = O_P(\varepsilon_{nT}).
$$

Then the first part of Theorem 1 follows.

Next, similar to the proof of Proposition 1 in Bai (2003) it can be shown that $\hat{\boldsymbol{H}} \to \boldsymbol{H} > 0$. Thus, $\hat{\boldsymbol{H}}$ is invertible with probability approaching 1. Note that $\hat{\boldsymbol{G}}(\boldsymbol{X}) = \hat{\boldsymbol{Y}}\hat{\boldsymbol{F}}/T = \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\hat{\boldsymbol{F}}/T + \hat{\boldsymbol{V}}\hat{\boldsymbol{F}}/T$. Write $\boldsymbol{F} = \hat{\boldsymbol{F}}\hat{\boldsymbol{H}}^{-1} + \boldsymbol{F} - \hat{\boldsymbol{F}}\hat{\boldsymbol{H}}^{-1}$, then

$$\hat{\boldsymbol{G}}(\boldsymbol{X}) = \boldsymbol{G}(\boldsymbol{X})(\hat{\boldsymbol{H}}')^{-1} + \boldsymbol{G}(\boldsymbol{X})(\boldsymbol{F} - \hat{\boldsymbol{F}}\hat{\boldsymbol{H}}^{-1})'\hat{\boldsymbol{F}}/T + \hat{\boldsymbol{V}}\hat{\boldsymbol{F}}/T,$$

and thus

$$\|\hat{\boldsymbol{G}}(\boldsymbol{X}) - \boldsymbol{G}(\boldsymbol{X})(\hat{\boldsymbol{H}}')^{-1}\|\sqrt{n} \leq \frac{\|\boldsymbol{G}(\boldsymbol{X})\|}{\sqrt{n}} \cdot \frac{\|\boldsymbol{F} - \hat{\boldsymbol{F}}\hat{\boldsymbol{H}}^{-1}\|}{\sqrt{T}} \cdot \frac{\|\hat{\boldsymbol{F}}\|}{\sqrt{T}} + \frac{\|\hat{\boldsymbol{V}}\|}{\sqrt{nT}} \cdot \frac{\|\hat{\boldsymbol{F}}\|}{\sqrt{T}} = O_P(\varepsilon_{nT}).$$

Then the second part of Theorem 1 follows.

Finally, note that $\hat{\boldsymbol{B}} = \hat{\boldsymbol{A}}\hat{\boldsymbol{F}}/T = \boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T) + (\hat{\boldsymbol{A}} - \boldsymbol{A}_0)\hat{\boldsymbol{F}}/T$. It follows from Proposition 1 that

$$\|\hat{\boldsymbol{B}} - \boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)\| \leq \frac{\|\hat{\boldsymbol{A}} - \boldsymbol{A}_0\|}{\sqrt{T}} \cdot \frac{\|\hat{\boldsymbol{F}}\|}{\sqrt{T}} = O_P(\varepsilon_{nT}). \tag{A.3}$$

Thus, for any $\boldsymbol{x} \in \mathcal{X}$,

$$\begin{aligned}\hat{\boldsymbol{g}}(\boldsymbol{x})' &= \boldsymbol{\phi}_{k_n}(\boldsymbol{x})'\hat{\boldsymbol{B}} = \boldsymbol{\phi}_{k_n}(\boldsymbol{x})'\boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T) + \boldsymbol{\phi}_{k_n}(\boldsymbol{x})'\left(\hat{\boldsymbol{B}} - \boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)\right)\\ &= \boldsymbol{g}(\boldsymbol{x})'(\hat{\boldsymbol{H}}^{-1})' + (\boldsymbol{\phi}_{k_n}(\boldsymbol{x})'\boldsymbol{B}_0 - \boldsymbol{g}(\boldsymbol{x})')(\boldsymbol{F}'\hat{\boldsymbol{F}}/T) + \boldsymbol{\phi}_{k_n}(\boldsymbol{x})'\left(\hat{\boldsymbol{B}} - \boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)\right) + O_P(\varepsilon_{nT}).\end{aligned}$$

Thus, it follows from (A.3) and Assumption 1 that

$$\sup_{\mathcal{X}} \left\|\hat{\boldsymbol{g}}(\boldsymbol{x}) - \hat{\boldsymbol{H}}^{-1}\boldsymbol{g}(\boldsymbol{x})\right\| \leq O_P(k_n^{-\alpha}) + \sup_{\mathcal{X}} \|\boldsymbol{\phi}_{k_n}(\boldsymbol{x})\| \cdot O_P(\varepsilon_{nT}) = O_P(\sqrt{k_n}\varepsilon_{nT}).$$

This completes the proof. □

Lemma 2. *Let $\xi_{it} = \theta_{0t}(\boldsymbol{x}_i) - \pi_n\theta_{0t}(\boldsymbol{x}_i) = \boldsymbol{g}(\boldsymbol{x}_i)'\boldsymbol{f}_t - \boldsymbol{a}_{0t}'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)$ and $\psi_{it} = \mathsf{F}(-\xi_{it}) - \mathbf{1}\{u_{it} \leq -\xi_{it}\}$. If Assumptions 1 to 3 hold, then*

$$\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t} - \mathsf{f}^{-1}(0) \cdot \hat{\boldsymbol{\Sigma}}_{\phi}^{-1} \cdot \frac{1}{n}\sum_{i=1}^{n}\psi_{it}\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\right\|^2} = O_P\left(k_n^{-\alpha}\right) + O_P\left(\eta_{nT}\right).$$

Proof. **Step 1:** For any $\boldsymbol{a} \in \mathbb{R}^{Dk_n}$ define:

$$\boldsymbol{m}_t(\boldsymbol{a}) = \frac{1}{n}\sum_{i=1}^{n}\left[\tau - \mathbf{1}\{u_{it} \leq (\boldsymbol{a} - \boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) - \xi_{it}\}\right]\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i),$$

$$\boldsymbol{m}_t^*(\boldsymbol{a}) = \frac{1}{n}\sum_{i=1}^n \left[\tau - \mathsf{F}\left((\boldsymbol{a}-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) - \xi_{it}\right)\right]\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i).$$

Since $\mathsf{F}(-\xi_{it}) = \tau - \mathsf{f}(-\xi_{it}^*)\cdot\xi_{it}$ where ξ_{it}^* is between 0 and ξ_{it}, it follows that

$$\boldsymbol{m}_t^*(\boldsymbol{a}_{0t}) = \frac{1}{n}\sum_{i=1}^n \mathsf{f}(-\xi_{it}^*)\cdot\xi_{it}\cdot\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i). \tag{A.4}$$

Taylor Expansion of $\boldsymbol{m}_t^*(\hat{\boldsymbol{a}}_t)$ around $\boldsymbol{a}_{0t}$ gives

$$\boldsymbol{m}_t^*(\hat{\boldsymbol{a}}_t) = \boldsymbol{m}_t^*(\boldsymbol{a}_{0t}) - \boldsymbol{M}_t^*(\tilde{\boldsymbol{a}}_t)\cdot(\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t}) \tag{A.5}$$

where $\tilde{\boldsymbol{a}}_t$ is between $\boldsymbol{a}_{0t}$ and $\hat{\boldsymbol{a}}_t$ and

$$\boldsymbol{M}_t^*(\tilde{\boldsymbol{a}}_t) = -\frac{\partial \boldsymbol{m}_t^*(\boldsymbol{a})}{\partial \boldsymbol{a}'}|_{\boldsymbol{a}=\tilde{\boldsymbol{a}}_t} = \frac{1}{n}\sum_{i=1}^n \mathsf{f}\left((\tilde{\boldsymbol{a}}_t-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) - \xi_{it}\right)\cdot\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)'. \tag{A.6}$$

By Assumption 3(ii) one can write

$$\boldsymbol{M}_t^*(\tilde{\boldsymbol{a}}_t) = \mathsf{f}(0)\cdot\hat{\boldsymbol{\Sigma}}_\phi + n^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{D}_t^*\boldsymbol{\Phi}(\boldsymbol{X}), \tag{A.7}$$

where $\hat{\boldsymbol{\Sigma}}_\phi = n^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})$ and $\boldsymbol{D}_t^*$ is a $n\times n$ diagonal matrix whose diagonal elements are bounded by in absolute values by $L\,|(\tilde{\boldsymbol{a}}_t-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) - \xi_{it}|$. Note that by Lemma 1,

$$\begin{aligned}\max_{1\le t\le T}\|\boldsymbol{D}_t^*\|_S \lesssim \max_{i,t}\left|(\tilde{\boldsymbol{a}}_t-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) - \xi_{it}\right| \\ \le \max_{1\le t\le T}\|\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t}\|\cdot O_P(\sqrt{k_n}) + O_P(k_n^{-\alpha}) = O_P(\sqrt{k_n}\varepsilon_{nT}).\end{aligned} \tag{A.8}$$

Moreover, one can write

$$\boldsymbol{m}_t^*(\hat{\boldsymbol{a}}_t) = \boldsymbol{m}_t(\hat{\boldsymbol{a}}_t) - \tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) + [\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) - \tilde{\boldsymbol{m}}_t(\hat{\boldsymbol{a}}_t)] \tag{A.9}$$

where $\tilde{\boldsymbol{m}}_t(\boldsymbol{a}) = \boldsymbol{m}_t(\boldsymbol{a}) - \boldsymbol{m}_t^*(\boldsymbol{a})$. It then follows from (A.5) (A.7) and (A.9) that

$$\begin{aligned}\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t} - \mathsf{f}^{-1}(0)\cdot\hat{\boldsymbol{\Sigma}}_\phi^{-1}\cdot\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) = \mathsf{f}^{-1}(0)\cdot\hat{\boldsymbol{\Sigma}}_\phi^{-1} \\ \left\{\boldsymbol{m}_t^*(\boldsymbol{a}_{0t}) - \boldsymbol{m}_t(\hat{\boldsymbol{a}}_t) - [\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) - \tilde{\boldsymbol{m}}_t(\hat{\boldsymbol{a}}_t)] - n^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{D}_t^*\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t})\right\},\end{aligned}$$

where

$$\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) = \frac{1}{n}\sum_{i=1}^n \left[\mathsf{F}(-\xi_{it}) - \mathbf{1}\{u_{it}\le -\xi_{it}\}\right]\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) = \frac{1}{n}\sum_{i=1}^n \psi_{it}\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i).$$

Since $\mathsf{f}(0)$ is bounded below, and $\lambda_{\min}(\hat{\boldsymbol{\Sigma}}_\phi)$ is bounded below with probability approaching 1, it suffices to show that

$$\max_{1\le t\le T}\|\boldsymbol{m}_t^*(\boldsymbol{a}_{0t})\| = O_P(k_n^{-\alpha}), \tag{A.10}$$

$$\max_{1\le t\le T}\|\boldsymbol{m}_t(\hat{\boldsymbol{a}}_t)\| = O_P(k_n^{3/2}/n), \tag{A.11}$$

$$\frac{1}{T}\sum_{t=1}^{T}\|\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) - \tilde{\boldsymbol{m}}_t(\hat{\boldsymbol{a}}_t)\|^2 = O_P\left(\eta_{nT}^2\right), \tag{A.12}$$

$$\max_{1\le t\le T}\left\|n^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{D}_t^*\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{a}}_t - \boldsymbol{a}_{0t})\right\| = O_P(\sqrt{k_n}\varepsilon_{nT}^2). \tag{A.13}$$

Step 2: By (A.4) and Assumption 1,

$$\begin{aligned}
&\max_{1\le t\le T}\|\boldsymbol{m}_t^*(\boldsymbol{a}_{0t})\| \\
= \quad &\max_{1\le t\le T}\left\|\frac{1}{n}\sum_{i=1}^{N}\mathsf{f}\left(-\xi_{it}^*\right)\cdot\xi_{it}\cdot\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\right\| \\
\le \quad &\max_{1\le t\le T}\left\|\frac{1}{n}\sum_{i=1}^{N}\mathsf{f}\left(0\right)\cdot\xi_{it}\cdot\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\right\| + O_P\left(k_n^{1/2-2\alpha}\right).
\end{aligned}$$

Define $z_{it} = \mathsf{f}(0)\cdot\xi_{it}$ and $\boldsymbol{z}_t = (z_{1t},\ldots,z_{Nt})'$, then

$$\frac{1}{n}\sum_{i=1}^{N}\mathsf{f}\left(0\right)\cdot\xi_{it}\cdot\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) = N^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{z}_t$$

and

$$\begin{aligned}
&\max_{1\le t\le T}\left\|\frac{1}{n}\sum_{i=1}^{N}\mathsf{f}\left(0\right)\cdot\xi_{it}\cdot\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\right\| \\
= \quad &\max_{1\le t\le T}\left\|N^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{z}_t\right\| \le \left\|N^{-1/2}\boldsymbol{\Phi}(\boldsymbol{X})\right\|_S\cdot\max_{1\le t\le T}\left\|N^{-1/2}\boldsymbol{z}_t\right\| = O_P(k_n^{-\alpha}).
\end{aligned}$$

In sum, we have

$$\max_{1\le t\le T}\|\boldsymbol{m}_t^*(\boldsymbol{a}_{0t})\| = O_P(k_n^{1/2-2\alpha}) + O_P(k_n^{-\alpha}) = O_P(k_n^{-\alpha}),$$

which gives (A.10).

Step 3: Similar to the proof of Lemma A4 of Horowitz and Lee (2005) it can be shown that

$$\max_{1\le t\le T}\|\boldsymbol{m}_t(\hat{\boldsymbol{a}}_t)\| = O_P(k_n^{3/2}/n),$$

which gives (A.11).

Step 4: By (A.8) and Lemma 1

$$\max_{1\le t\le T}\left\|n^{-1}\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{D}_t^*\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{a}}_t-\boldsymbol{a}_{0t})\right\| \\ \le \|\boldsymbol{\Phi}(\boldsymbol{X})/\sqrt{n}\|_S^2\cdot\max_{1\le t\le T}\|\boldsymbol{D}_t^*\|_S\cdot\max_{1\le t\le T}\|\hat{\boldsymbol{a}}_t-\boldsymbol{a}_{0t}\| = O_P(\sqrt{k_n}\varepsilon_{nT}^2),$$

which gives (A.13).

Step 5: Define:

$$\delta_{1t}(\boldsymbol{\alpha}) = \frac{1}{n}\sum_{i=1}^n\left[\mathbf{1}\{u_{it}\le(\boldsymbol{a}-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)-\xi_{it}\}-\mathbf{1}\{u_{it}\le-\xi_{it}\}\right]\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i),$$

$$\delta_{2t}(\boldsymbol{\alpha}) = \frac{1}{n}\sum_{i=1}^n\left[\mathsf{F}\left((\boldsymbol{a}-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)-\xi_{it}\right)-\mathsf{F}\left(-\xi_{it}\right)\right]\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i),$$

$$\tilde{\delta}_{1t}(\boldsymbol{\alpha}) = \delta_{1t}(\boldsymbol{\alpha})-\mathbb{E}[\delta_{1t}(\boldsymbol{\alpha})], \qquad \tilde{\delta}_{2t}(\boldsymbol{\alpha}) = \delta_{2t}(\boldsymbol{\alpha})-\mathbb{E}[\delta_{2t}(\boldsymbol{\alpha})].$$

Note that $\mathbb{E}[\delta_{1t}(\boldsymbol{\alpha})] = \mathbb{E}[\delta_{2t}(\boldsymbol{\alpha})]$ because $\delta_{2t}(\boldsymbol{\alpha}) = \mathbb{E}[\delta_{1t}(\boldsymbol{\alpha})|\boldsymbol{x}_i]$. Then $\tilde{\boldsymbol{m}}_t(\hat{\boldsymbol{a}}_t)-\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t}) = \tilde{\delta}_{2t}(\hat{\boldsymbol{a}}_t)-\tilde{\delta}_{1t}(\hat{\boldsymbol{a}}_t)$, and

$$\frac{1}{T}\sum_{t=1}^T\|\tilde{\boldsymbol{m}}_t(\hat{\boldsymbol{a}}_t)-\tilde{\boldsymbol{m}}_t(\boldsymbol{a}_{0t})\|^2 \le \frac{1}{T}\sum_{t=1}^T\left\|\tilde{\delta}_{1t}(\hat{\boldsymbol{a}}_t)\right\|^2+\frac{1}{T}\sum_{t=1}^T\left\|\tilde{\delta}_{2t}(\hat{\boldsymbol{a}}_t)\right\|^2. \tag{A.14}$$

In what follows, we will show that

$$\frac{1}{T}\sum_{t=1}^T\left\|\tilde{\delta}_{1t}(\hat{\boldsymbol{a}}_t)\right\|^2 = O_P\left(\ln(k_n^{-1/4}\varepsilon_{nT}^{-1/2})\cdot k_n^{5/2}\varepsilon_{nT}n^{-1}\right), \tag{A.15}$$

$$\frac{1}{T}\sum_{t=1}^T\left\|\tilde{\delta}_{2t}(\hat{\boldsymbol{a}}_t)\right\|^2 = O_P\left(\ln(k_n^{-1/2}\varepsilon_{nT}^{-1})\cdot k_n^{3}\varepsilon_{nT}^2n^{-1}\right), \tag{A.16}$$

which imply (A.12) and therefore complete the proof. We will focus on the proof of (A.15) since the proof of (A.16) is similar.

Let $\phi_{jd}(\boldsymbol{x}_i)$ be the jdth element of $\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)$ for $j=1,\dots,k_n; d=1,\dots,D$, and define

$$\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i) = \mathbf{1}\{u_{it}\le(\boldsymbol{a}-\boldsymbol{a}_{0t})'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)-\xi_{it}\}-\mathbf{1}\{u_{it}\le-\xi_{it}\}.$$

Then for some $C>0$, with probability approach 1,

$$\frac{1}{T}\sum_{t=1}^T\left\|\tilde{\delta}_{1t}(\hat{\boldsymbol{a}}_t)\right\|^2 \le \frac{1}{n}\cdot\frac{1}{T}\sum_{t=1}^T\sum_{j=1}^{k_n}\sum_{d=1}^D\sup_{\|\boldsymbol{a}-\boldsymbol{a}_{0t}\|\le C\varepsilon_{nT}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n\{\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)-\mathbb{E}[\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)]\}\right|^2$$

We will show that

$$\mathbb{E}\left[\sup_{\|\boldsymbol{a}-\boldsymbol{a}_{0t}\|\le C\varepsilon_{nT}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)-\mathbb{E}[\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)]\}\right|^2\right]$$
$$= O\left(\ln(k_n^{-1/4}\varepsilon_{nT}^{-1/2})\cdot k_n^{3/2}\varepsilon_{nT}\right) \quad \text{(A.17)}$$

uniformly in t and j, from which (A.15) follows.

Define $\mathcal{H}_{\varepsilon_{nT}} = \{h(\boldsymbol{a},\boldsymbol{x}_i) \equiv \Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i) - \mathbb{E}[\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)] : \|\boldsymbol{a}-\boldsymbol{a}_{0t}\| \le C\varepsilon_{nT}\}$, and for any $h \in \mathcal{H}_{\varepsilon_{nT}}$ define $\mathbb{G}_n h = n^{-1/2}\sum_{i=1}^{n} h(\boldsymbol{a},\boldsymbol{x}_i)$. Write

$$\sup_{\|\boldsymbol{a}-\boldsymbol{a}_{0t}\|\le C\varepsilon_{n}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)-\mathbb{E}[\Delta_{it}(\boldsymbol{\alpha},\boldsymbol{x}_i)\phi_{jd}(\boldsymbol{x}_i)]\}\right| = \|\mathbb{G}_n h\|_{\mathcal{H}_{\varepsilon_{nT}}},$$

then the left-hand side of (A.17) can be written as $\mathbb{E}\,\|\mathbb{G}_n h\|^2_{\mathcal{H}_{\varepsilon_{nT}}}$. Let $N(\mathcal{H}_{\varepsilon_{nT}}, L_2(Q), \epsilon)$ be the covering number of $\mathcal{H}_{\varepsilon_{nT}}$, where $L_2(Q)$ is the L_2 norm for functions and Q is any probability measure on $\mathcal{X}$. Similar to the proof of (A.12) in Kato et al. (2012), it can be shown that $N(\mathcal{H}_{\varepsilon_{nT}}, L_2(Q), 2\epsilon) \le (A/\epsilon)^{c_1 k_n}$ for some bounded constant c_1 and $A \ge 3\sqrt{e}$ that do not depend on t and j. Moreover, it is easy to show that $\sup_{h\in\mathcal{H}_{\varepsilon_{nT}}}\mathbb{E}[h^2(\boldsymbol{a},\boldsymbol{x}_i)] \le c_2^2\sqrt{k_n}\varepsilon_n$ for some bounded constant c_2. Then, applying Proposition B.1 of Kato et al. (2012), we have

$$\mathbb{E}\,\|\mathbb{G}_n h\|_{\mathcal{H}_{\varepsilon_{nT}}} \le c_3\left[\cdot\ln(c_4 k_n^{-1/4}\varepsilon_{nT}^{-1/2})\cdot k_n/\sqrt{n} + \sqrt{\ln(c_4 k_n^{-1/4}\varepsilon_{nT}^{-1/2})}\cdot k_n^{3/4}\varepsilon_{nT}^{1/2}\right]$$
$$\le c_5\sqrt{\ln(k_n^{-1/4}\varepsilon_{nT}^{-1/2})}\cdot k_n^{3/4}\varepsilon_{nT}^{1/2}, \quad \text{(A.18)}$$

where c_3, c_4, c_5 are bounded constants that do not depend on t and j. Finally, (A.17) follows by noting that (see Chapter 6 of Ledoux and Talagrand 1991)

$$\mathbb{E}\,\|\mathbb{G}_n h\|^2_{\mathcal{H}_{\varepsilon_{nT}}} \le \left(\mathbb{E}\,\|\mathbb{G}_n h\|_{\mathcal{H}_{\varepsilon_{nT}}}\right)^2 + O(n^{-1}).$$

This completes the proof. □

Proof of Theorem 2:

Proof. Let $\boldsymbol{\Psi}$ be the $n \times T$ matrix of ψ_{it}, then the result of Lemma 2 can be written as

$$\left\|\hat{\boldsymbol{A}} - \boldsymbol{A}_0 - \mathsf{f}(0)^{-1}\cdot\hat{\boldsymbol{\Sigma}}_\phi^{-1}\boldsymbol{\Phi}'(\boldsymbol{X})\boldsymbol{\Psi}/n\right\|/\sqrt{T} = O_P\left(k_n^{-\alpha}\right) + O_P\left(\eta_{nT}\right). \quad \text{(A.19)}$$

From (A.2) and Lemma 1 we have

$$\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T} \leq O_P(1) \cdot \|\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{V}}/(nT)\|_S + O_P(\varepsilon_{nT}^2). \tag{A.20}$$

Define $\boldsymbol{R}(\boldsymbol{X}) = \boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{B}_0 - \boldsymbol{G}(\boldsymbol{X})$, then by Assumption 1(ii) $\|\boldsymbol{R}(\boldsymbol{X})\|/\sqrt{n} = O_P(k_n^{-\alpha})$. Moreover, we can write

$$\begin{aligned}
\hat{\boldsymbol{V}} &= \hat{\boldsymbol{Y}} - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}' \\
&= \boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}} - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}' \\
&= \boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}} - \boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{A}_0 + \boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{A}_0 - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}' \\
&= \boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0) + \boldsymbol{R}(\boldsymbol{X})\boldsymbol{F}'.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{V}}/(nT) \\
=\ &\boldsymbol{F}(\boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{B}_0 - \boldsymbol{R}(\boldsymbol{X}))'[\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0) + \boldsymbol{R}(\boldsymbol{X})\boldsymbol{F}']/(nT) \\
=\ &\boldsymbol{F}\boldsymbol{B}_0'\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0)/(nT) - \boldsymbol{F}\boldsymbol{R}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0)/(nT) \\
&+\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\boldsymbol{R}(\boldsymbol{X})\boldsymbol{F}'/(nT).
\end{aligned}$$

It then follows from Theorem 1 and Lemma 1 that

$$\|\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{V}}/(nT)\|_S \leq \|\boldsymbol{F}\boldsymbol{B}_0'\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0)/(nT)\|_S + O_P(k_n^{-\alpha}).$$

The above inequality and (A.20) imply that

$$\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T} \leq \|\boldsymbol{F}\boldsymbol{B}_0'\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0)/(nT)\|_S + O_P(k_n^{-\alpha}) + O_P(\varepsilon_{nT}^2). \tag{A.21}$$

By (A.19) and Assumption 1(ii), we have

$$\begin{aligned}
&\|\boldsymbol{F}\boldsymbol{B}_0'\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}} - \boldsymbol{A}_0)/(nT)\|_S \\
\leq\ &\mathsf{f}(0)^{-1}\|\boldsymbol{B}_0'\boldsymbol{\Phi}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{\Sigma}}_\phi^{-1}\boldsymbol{\Phi}'(\boldsymbol{X})\boldsymbol{\Psi}/(n^2T^{1/2})\|_S + O_P\left(k_n^{-\alpha} + \eta_{nT}\right) \\
=\ &\mathsf{f}(0)^{-1}\|\boldsymbol{B}_0'\boldsymbol{\Phi}'(\boldsymbol{X})\boldsymbol{\Psi}/(nT^{1/2})\|_S + O_P\left(k_n^{-\alpha} + \eta_{nT}\right) \\
\leq\ &\mathsf{f}(0)^{-1}\|\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{\Psi}/(nT^{1/2})\| + \|\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{B}_0\|/\sqrt{n} \cdot \|\boldsymbol{\Psi}\|/\sqrt{nT} + O_P\left(k_n^{-\alpha} + \eta_{nT}\right) \\
=\ &\mathsf{f}(0)^{-1}\|\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{\Psi}/(nT^{1/2})\| + O_P\left(k_n^{-\alpha} + \eta_{nT}\right).
\end{aligned}$$

Note that

$$\|\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{\Psi}/(nT^{1/2})\| = \frac{1}{\sqrt{n}} \cdot \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{g}(\boldsymbol{x}_i)\psi_{it}\right\|^2} = O_P(n^{-1/2})$$

because it is easy to see that $\mathbb{E}\left\|n^{-1/2}\sum_{i=1}^{n}\boldsymbol{g}(\boldsymbol{x}_i)\psi_{it}\right\|^2<\infty$ for all t. It then follows from (A.21) that

$$\|\hat{\boldsymbol{F}}-\boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T}=O_P(n^{-1/2})+O_P(k_n^{-\alpha})+O_P(\eta_{nT})+O_P(\varepsilon_{nT}^2).$$

This completes the proof. □

Lemma 3. *Under Assumptions 1, 2 and 4, we have*

$$\left\|\hat{\boldsymbol{A}}-\boldsymbol{A}_0-\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\Phi}'(\boldsymbol{X})\boldsymbol{\Psi}(\boldsymbol{X})/n\right\|/\sqrt{T}=O_P\left(k_n^{-\alpha}\right)+O_P\left(\eta_{nT}\right).$$

where $\psi_{it}(\boldsymbol{x}_i)=\mathsf{F}\left(-\xi_{it}|\boldsymbol{x}_i\right)-\mathbf{1}\{u_{it}\le-\xi_{it}\}$ *and* $\boldsymbol{\Psi}(\boldsymbol{X})$ *is the* $n\times T$ *matrix of* $\psi_{it}(\boldsymbol{x}_i)$.

Proof. The proof is similar to the proof of Lemma 2. Therefore, it is omitted to save space. □

Proof of Theorem 3:

Proof. By the proof of Theorem 1, for any $\boldsymbol{x}\in\mathcal{X}$,

$$\hat{\boldsymbol{g}}(\boldsymbol{x})=(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'\boldsymbol{g}(\boldsymbol{x})+(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'(\boldsymbol{B}_0'\boldsymbol{\phi}_{k_n}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{x}))+(\hat{\boldsymbol{B}}-\boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T))'\boldsymbol{\phi}_{k_n}(\boldsymbol{x}).$$

Moreover,

$$\hat{\boldsymbol{B}}-\boldsymbol{B}_0(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)=(\hat{\boldsymbol{A}}-\boldsymbol{A}_0)\boldsymbol{F}\hat{\boldsymbol{H}}/T+(\hat{\boldsymbol{A}}-\boldsymbol{A}_0)(\hat{\boldsymbol{F}}-\boldsymbol{F}\hat{\boldsymbol{H}})/T.$$

Thus, by Lemma 1 and Theorem 1,

$$\hat{\boldsymbol{g}}(\boldsymbol{x})-(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'\boldsymbol{g}(\boldsymbol{x})=\hat{\boldsymbol{H}}'\boldsymbol{F}'(\hat{\boldsymbol{A}}-\boldsymbol{A}_0)'\boldsymbol{\phi}_{k_n}(\boldsymbol{x})/T+O_P(k_n^{-\alpha})+O_P(\varepsilon_n^2\sqrt{k_n}).$$

It then follows from Lemma 3 that

$$\hat{\boldsymbol{g}}(\boldsymbol{x})-(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'\boldsymbol{g}(\boldsymbol{x})=\hat{\boldsymbol{H}}'\boldsymbol{F}'\boldsymbol{\Psi}'(\boldsymbol{X})\boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\phi}_{k_n}(\boldsymbol{x})/(nT)+O_P(k_n^{1/2-\alpha})+O_P(\sqrt{k_n}\eta_{nT}).$$

Define $\boldsymbol{d}_T(\boldsymbol{x}_i)=T^{-1}\sum_{t=1}^{T}\boldsymbol{f}_t\psi_{it}(\boldsymbol{x}_i)$, $q(\boldsymbol{x}_i)=\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)'\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\phi}_{k_n}(\boldsymbol{x})$, then we can write

$$\boldsymbol{F}'\boldsymbol{\Psi}'(\boldsymbol{X})\boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\phi}_{k_n}(\boldsymbol{x})/(nT)=\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{d}_T(\boldsymbol{x}_i)q(\boldsymbol{x}_i).$$

Note that $\mathbb{E}[\boldsymbol{d}_T(\boldsymbol{x}_i)q(\boldsymbol{x}_i)]=0$ because $\mathbb{E}[\boldsymbol{d}_T(\boldsymbol{x}_i)|\boldsymbol{x}_i]=0$, and it is easy to show that

$$\begin{aligned}\mathbb{E}[\boldsymbol{d}_T(\boldsymbol{x}_i)\boldsymbol{d}_T(\boldsymbol{x}_i)'q^2(\boldsymbol{x}_i)]&=\tau(1-\tau)(\boldsymbol{F}'\boldsymbol{F}/T^2)\boldsymbol{\phi}_{k_n}'(\boldsymbol{x})\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\Sigma}_{\phi}\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\boldsymbol{\phi}_{k_n}(\boldsymbol{x})+o(1)\\&=\tau(1-\tau)(\boldsymbol{F}'\boldsymbol{F}/T^2)\sigma_{k_n}^2+o(1).\end{aligned}$$

Thus, we have

$$\boldsymbol{\Sigma}_{T,\tau}^{-1/2}(\hat{\boldsymbol{H}}')^{-1}\cdot\frac{\sqrt{nT}}{\sigma_{k_n}}\Big(\hat{\boldsymbol{g}}(\boldsymbol{x})-(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'\boldsymbol{g}(\boldsymbol{x})\Big)=\boldsymbol{\Sigma}_{T,\tau}^{-1/2}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sqrt{T}\boldsymbol{d}_T(\boldsymbol{x}_i)q(\boldsymbol{x}_i)/\sigma_{k_n}$$
$$+O_P(k_n^{1/2-\alpha}+\sqrt{k_n}\eta_{nT})\sqrt{nT}\sigma_{k_n}^{-1}. \quad \text{(A.22)}$$

Finally, it follows from the Lyapunov's CLT and Assumption 4(iv) that

$$\boldsymbol{\Sigma}_{T,\tau}^{-1/2}(\hat{\boldsymbol{H}}')^{-1}\cdot\frac{\sqrt{nT}}{\sigma_{k_n}}\Big(\hat{\boldsymbol{g}}(\boldsymbol{x})-(\boldsymbol{F}'\hat{\boldsymbol{F}}/T)'\boldsymbol{g}(\boldsymbol{x})\Big)\xrightarrow{d}N(0,\boldsymbol{I}_R).$$

This completes the proof. □

Proof of Theorem 4:

Proof. Define $\boldsymbol{R}(\boldsymbol{X})=\boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{B}_0-\boldsymbol{G}(\boldsymbol{X})$, we can write

$$\hat{\boldsymbol{Y}}=\boldsymbol{\Phi}(\boldsymbol{X})\boldsymbol{A}_0+\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}}-\boldsymbol{A}_0)=\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'+\boldsymbol{R}(\boldsymbol{X})\boldsymbol{F}'+\boldsymbol{\Phi}(\boldsymbol{X})(\hat{\boldsymbol{A}}-\boldsymbol{A}_0).$$

Thus,

$$\tilde{\boldsymbol{F}}=\hat{\boldsymbol{Y}}'\hat{\boldsymbol{G}}(\boldsymbol{X})\cdot(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X}))^{-1}=\boldsymbol{F}(\boldsymbol{G}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}$$
$$+\boldsymbol{F}(\boldsymbol{R}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}+(\hat{\boldsymbol{A}}-\boldsymbol{A}_0)'(\boldsymbol{\Phi}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1},$$

and

$$\tilde{\boldsymbol{f}}_t-\tilde{\boldsymbol{H}}'\boldsymbol{f}_t=(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}(\hat{\boldsymbol{G}}(\boldsymbol{X})'\boldsymbol{R}(\boldsymbol{X})/n)\boldsymbol{f}_t$$
$$+(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}(\hat{\boldsymbol{G}}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})/n)(\hat{\boldsymbol{a}}_t-\boldsymbol{a}_{0t}).$$

It is easy to see from Theorem 1 and Assumption 1(ii) that the first term on the right-hand side of the above equation is $O_P(k_n^{-\alpha})$. Moreover, by Lemma 3, the second term can be written as

$$(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}\cdot(\hat{\boldsymbol{G}}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})/n)\cdot\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\cdot\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\psi_{it}(\boldsymbol{x}_i)+O_P(k_n^{-\alpha})+O_P(\eta_{nT}).$$

By Theorem 1 we can show that

$$\|(\hat{\boldsymbol{G}}(\boldsymbol{X})'\hat{\boldsymbol{G}}(\boldsymbol{X})/n)^{-1}-\hat{\boldsymbol{H}}'\boldsymbol{\Sigma}_g^{-1}\hat{\boldsymbol{H}}\|=O_P(\varepsilon_{nT}),$$

$$\|(\hat{\boldsymbol{G}}(\boldsymbol{X})'\boldsymbol{\Phi}(\boldsymbol{X})/n)-\hat{\boldsymbol{H}}^{-1}\mathbb{E}[\boldsymbol{g}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)']\|_S=O_P(\varepsilon_{nT}),$$

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i) \psi_{it}(\boldsymbol{x}_i) \right\| = O_P(\sqrt{k_n/n}),$$

it then follows from Assumption 4(iii) that

$$(\hat{\boldsymbol{H}}')^{-1}\sqrt{n}(\tilde{\boldsymbol{f}}_t - \tilde{\boldsymbol{H}}'\boldsymbol{f}_t) = \boldsymbol{\Sigma}_g^{-1}\mathbb{E}[\boldsymbol{g}(\boldsymbol{x}_i)\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)']\boldsymbol{\Sigma}_{\mathsf{f}\phi}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\psi_{it}(\boldsymbol{x}_i)\right) \\ + O_P(\varepsilon_{nT}k_n^{1/2}) + O_P(n^{1/2}k_n^{-\alpha}) + O_P(n^{1/2}\eta_{nT}).$$

By the Lyapunov's CLT we can show that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{\phi}_{k_n}(\boldsymbol{x}_i)\psi_{it}(\boldsymbol{x}_i) \xrightarrow{d} N(0, \tau(1-\tau)\boldsymbol{\Sigma}_\phi),$$

then the desired result follows from Assumption 5. □

Proof of Theorem 5:

Proof. First, note that

$$\begin{aligned} & \|\boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}'\boldsymbol{\Phi}(\boldsymbol{X})' - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\|/(nT) \\ \leq\ & 2\|\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\|/\sqrt{nT}\cdot\|\boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}} - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\|/\sqrt{nT} + \|\boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}} - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\|^2/(nT) \\ =\ & O_P(1)\cdot\|\hat{\boldsymbol{V}}\|/\sqrt{nT} + \|\hat{\boldsymbol{V}}\|^2/(nT). \end{aligned}$$

It then follows from Lemma 1(ii) that

$$\|\boldsymbol{\Phi}(\boldsymbol{X})\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}'\boldsymbol{\Phi}(\boldsymbol{X})' - \boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'\|/(nT) = O_P(\varepsilon_{nT}). \tag{A.23}$$

Second, Assumption 2(iii) and (iv) imply that the largest R eigenvalues of $\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'/(nT)$, which are also the R eigenvalues of $(\boldsymbol{F}'\boldsymbol{F}/T)\cdot\boldsymbol{G}(\boldsymbol{X})'\boldsymbol{G}(\boldsymbol{X})/n$, converge in probability to the R eigenvalues of $(\boldsymbol{F}'\boldsymbol{F}/T)\cdot\boldsymbol{\Sigma}_g$. Also, note that the remaining eigenvalues of $\boldsymbol{G}(\boldsymbol{X})\boldsymbol{F}'\boldsymbol{F}\boldsymbol{G}(\boldsymbol{X})'/(nT)$ are all 0, it then follows from (A.23) and the Wielandt-Hoffman inequality that $\hat{\rho}_j = O_P(\varepsilon_{nT})$ for $j = R+1, \ldots, \bar{R}$, and $\hat{\rho}_j$ converges in probability in some positive constant for $j = 1, \ldots, R$. The desired result then follows because $P[\hat{\rho}_j > p_n] \to 1$ for $j = 1, \ldots, R$ and $P[\hat{\rho}_j > p_n] \to 0$ for $j = R+1, \ldots, \bar{R}$. □

B Figures and Tables

Table 1: Estimating the number of factors

	T	n	$N(0,1)$			$T(3)$			Cauchy(0,1)		
$\tau = 0.25$	5	50	[0.13	0.65	0.23]	[0.03	0.41	0.56]	[0.01	0.10	0.89]
	5	100	[0.10	0.72	0.19]	[0.02	0.44	0.54]	[0.00	0.03	0.97]
	5	200	[0.23	0.77	0.00]	[0.12	0.82	0.06]	[0.00	0.17	0.83]
	5	1000	[0.17	0.83	0.00]	[0.16	0.84	0.00]	[0.06	0.81	0.13]
	10	50	[0.17	0.76	0.07]	[0.03	0.50	0.47]	[0.02	0.06	0.92]
	10	100	[0.08	0.89	0.03]	[0.03	0.65	0.46]	[0.00	0.03	0.97]
	10	200	[0.07	0.93	0.00]	[0.05	0.95	0.00]	[0.00	0.24	0.76]
	10	1000	[0.03	0.97	0.00]	[0.02	0.98	0.00]	[0.01	0.98	0.01]
$\tau = 0.5$	5	50	[0.19	0.71	0.10]	[0.09	0.56	0.35]	[0.00	0.15	0.85]
	5	100	[0.17	0.76	0.08]	[0.07	0.59	0.34]	[0.00	0.20	0.80]
	5	200	[0.23	0.77	0.00]	[0.19	0.80	0.01]	[0.06	0.75	0.19]
	5	1000	[0.18	0.82	0.00]	[0.15	0.85	0.00]	[0.13	0.87	0.00]
	10	50	[0.20	0.78	0.03]	[0.08	0.76	0.15]	[0.00	0.13	0.87]
	10	100	[0.12	0.87	0.01]	[0.05	0.87	0.08]	[0.00	0.24	0.76]
	10	200	[0.05	0.95	0.00]	[0.05	0.95	0.00]	[0.03	0.94	0.03]
	10	1000	[0.01	0.99	0.00]	[0.02	0.98	0.00]	[0.02	0.99	0.00]
$\tau = 0.75$	5	50	[0.11	0.68	0.21]	[0.04	0.41	0.56]	[0.01	0.09	0.90]
	5	100	[0.10	0.71	0.19]	[0.02	0.42	0.56]	[0.00	0.04	0.96]
	5	200	[0.22	0.78	0.00]	[0.14	0.81	0.05]	[0.00	0.15	0.85]
	5	1000	[0.18	0.82	0.00]	[0.17	0.83	0.00]	[0.04	0.82	0.15]
	10	50	[0.15	0.78	0.08]	[0.04	0.50	0.46]	[0.01	0.05	0.94]
	10	100	[0.11	0.86	0.04]	[0.03	0.65	0.32]	[0.00	0.03	0.97]
	10	200	[0.06	0.94	0.00]	[0.05	0.94	0.01]	[0.01	0.27	0.73]
	10	1000	[0.02	0.98	0.00]	[0.02	0.98	0.00]	[0.02	0.97	0.01]

Note: the DGP is $y_{it} = \sum_{r=1}^{3} \lambda_{ir} f_{tr} + \left(x_{i1}^2 + x_{i2}^2 + x_{i3}^2\right) u_{it}$, where $f_{t1} = 1$, $f_{t2}, f_{t3} \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} are drawn independently from the uniform distribution: $U[-1,1]$. $g_1(z) = sin(2\pi z)$, $g_2(z) = sin(\pi z)$ and $g_3(z) = cos(\pi z)$, and $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$, $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. u_{it} are i.i.d variables drawn from three different distributions. In the first step quantile sieve estimation, $k_n = n^{1/3}$ and we use the *Chebyshev polynomials of the second kind* as the basis functions. For the estimator of the number of factors, the threshold p_n is chosen as in (9) with $d = 1/4$. The reported results are [frequency of $\hat{R} < R$; frequency of $\hat{R} = R$; frequency of $\hat{R} > R$] from 1000 replications.

Table 2: Estimating the number of factors: eigen-ratio estimator

	T	n	$N(0,1)$			$T(3)$			Cauchy(0,1)		
$\tau = 0.25$	5	50	[0.57	0.25	0.19]	[0.54	0.22	0.25]	[0.54	0.17	0.29]
	5	100	[0.58	0.33	0.09]	[0.58	0.27	0.15]	[0.59	0.15	0.26]
	5	200	[0.44	0.54	0.01]	[0.54	0.43	0.04]	[0.62	0.24	0.14]
	5	1000	[0.23	0.77	0.00]	[0.31	0.69	0.00]	[0.56	0.42	0.02]
	10	50	[0.46	0.37	0.17]	[0.45	0.18	0.37]	[0.47	0.07	0.46]
	10	100	[0.37	0.59	0.04]	[0.46	0.42	0.11]	[0.60	0.09	0.31]
	10	200	[0.09	0.91	0.00]	[0.19	0.80	0.01]	[0.59	0.31	0.11]
	10	1000	[0.01	0.99	0.00]	[0.03	0.97	0.00]	[0.17	0.83	0.00]
$\tau = 0.5$	5	50	[0.58	0.28	0.14]	[0.57	0.22	0.20]	[0.50	0.20	0.30]
	5	100	[0.58	0.33	0.09]	[0.57	0.28	0.15]	[0.56	0.21	0.22]
	5	200	[0.42	0.57	0.01]	[0.46	0.51	0.03]	[0.54	0.41	0.06]
	5	1000	[0.21	0.79	0.00]	[0.23	0.77	0.00]	[0.28	0.72	0.00]
	10	50	[0.41	0.46	0.13]	[0.46	0.33	0.21]	[0.42	0.10	0.48]
	10	100	[0.30	0.66	0.04]	[0.36	0.57	0.07]	[0.51	0.24	0.26]
	10	200	[0.06	0.94	0.00]	[0.11	0.89	0.00]	[0.22	0.76	0.02]
	10	1000	[0.01	0.99	0.00]	[0.02	0.98	0.00]	[0.03	0.97	0.00]
$\tau = 0.75$	5	50	[0.58	0.25	0.17]	[0.54	0.22	0.24]	[0.55	0.17	0.28]
	5	100	[0.57	0.32	0.10]	[0.59	0.24	0.17]	[0.56	0.20	0.24]
	5	200	[0.43	0.55	0.02]	[0.52	0.43	0.04]	[0.65	0.21	0.14]
	5	1000	[0.24	0.76	0.00]	[0.33	0.67	0.00]	[0.55	0.44	0.01]
	10	50	[0.46	0.36	0.18]	[0.44	0.20	0.37]	[0.47	0.05	0.48]
	10	100	[0.36	0.59	0.06]	[0.46	0.40	0.14]	[0.63	0.09	0.28]
	10	200	[0.11	0.89	0.00]	[0.19	0.80	0.01]	[0.58	0.31	0.11]
	10	1000	[0.01	0.99	0.00]	[0.03	0.97	0.00]	[0.16	0.83	0.01]

Note: the DGP is $y_{it} = \sum_{r=1}^{3} \lambda_{ir} f_{tr} + \left(x_{i1}^2 + x_{i2}^2 + x_{i3}^2\right) u_{it}$, where $f_{t1} = 1$, $f_{t2}, f_{t3} \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} are drawn independently from the uniform distribution: $U[-1,1]$. $g_1(z) = sin(2\pi z)$, $g_2(z) = sin(\pi z)$ and $g_3(z) = cos(\pi z)$, and $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$, $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. u_{it} are i.i.d variables drawn from three different distributions. In the first step quantile sieve estimation, $k_n = n^{1/3}$ and we use the *Chebyshev polynomials of the second kind* as the basis functions. The estimator for the number of factors is the integer that maximizes the eigen-ratios. The reported results are [frequency of $\hat{R} < R$; frequency of $\hat{R} = R$; frequency of $\hat{R} > R$] from 1000 replications.

Table 3: Factor estimation using QPCA

	T	n	$N(0,1)$ f_{1t}	f_{2t}	f_{3t}	$T(3)$ f_{1t}	f_{2t}	f_{3t}	Cauchy(0,1) f_{1t}	f_{2t}	f_{3t}
$\tau = 0.25$	10	50	0.859	0.879	0.574	0.738	0.745	0.630	0.386	0.370	0.609
	10	100	0.971	0.956	0.857	0.938	0.890	0.835	0.670	0.566	0.767
	10	200	0.989	0.983	0.924	0.978	0.959	0.911	0.862	0.767	0.867
	10	500	0.997	0.995	0.979	0.994	0.990	0.972	0.968	0.940	0.950
	50	50	0.893	0.909	0.417	0.751	0.796	0.499	0.086	0.069	0.375
	50	100	0.976	0.968	0.824	0.957	0.940	0.797	0.623	0.407	0.654
	50	200	0.990	0.986	0.901	0.982	0.977	0.892	0.919	0.838	0.821
	50	500	0.997	0.995	0.973	0.995	0.992	0.967	0.984	0.975	0.941
$\tau = 0.75$	10	50	0.861	0.876	0.581	0.749	0.749	0.623	0.383	0.362	0.605
	10	100	0.971	0.955	0.858	0.933	0.894	0.834	0.682	0.573	0.768
	10	200	0.989	0.983	0.921	0.979	0.960	0.905	0.867	0.777	0.867
	10	500	0.997	0.995	0.979	0.994	0.990	0.974	0.973	0.937	0.950
	50	50	0.893	0.911	0.420	0.749	0.794	0.493	0.081	0.066	0.380
	50	100	0.977	0.967	0.824	0.958	0.938	0.794	0.617	0.400	0.656
	50	200	0.990	0.986	0.901	0.982	0.976	0.894	0.915	0.832	0.818
	50	500	0.997	0.995	0.972	0.995	0.992	0.967	0.984	0.974	0.938

Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2, 3, 4, 5$) are independently drawn from the uniform distribution: $U[-1,1]$. $g_1(x) = sin(2\pi x)$, $g_2(x) = sin(\pi x)$ *and* $g_3(x) = |cos(\pi x)|$. The factor loading functions are generated as $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ and $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. 3 factors are estimated at each τ using the proposed method in this paper, and the reported results are the averages of the adjusted R^2 of regressing the true factors on the estimated factors from 1000 replications.

Table 4: Factor estimation using QFA

	T	n	$N(0,1)$ f_{1t}	$N(0,1)$ f_{2t}	$N(0,1)$ f_{3t}	$T(3)$ f_{1t}	$T(3)$ f_{2t}	$T(3)$ f_{3t}	Cauchy(0,1) f_{1t}	Cauchy(0,1) f_{2t}	Cauchy(0,1) f_{3t}
$\tau = 0.25$	10	50	0.887	0.821	0.561	0.808	0.706	0.528	0.516	0.418	0.449
	10	100	0.898	0.833	0.586	0.822	0.727	0.574	0.525	0.427	0.501
	10	200	0.904	0.841	0.624	0.834	0.735	0.584	0.525	0.443	0.504
	10	500	0.908	0.840	0.643	0.841	0.740	0.608	0.513	0.420	0.512
	50	50	0.964	0.948	0.786	0.935	0.902	0.725	0.724	0.537	0.473
	50	100	0.983	0.976	0.884	0.972	0.956	0.848	0.871	0.767	0.669
	50	200	0.992	0.988	0.936	0.986	0.977	0.911	0.935	0.853	0.802
	50	500	0.996	0.994	0.965	0.994	0.989	0.951	0.963	0.906	0.880
$\tau = 0.75$	10	50	0.875	0.835	0.551	0.808	0.719	0.523	0.510	0.414	0.447
	10	100	0.898	0.938	0.595	0.820	0.730	0.583	0.523	0.420	0.506
	10	200	0.904	0.846	0.616	0.828	0.736	0.600	0.520	0.429	0.497
	10	500	0.899	0.838	0.625	0.843	0.742	0.616	0.528	0.433	0.489
	50	50	0.964	0.947	0.785	0.935	0.901	0.722	0.722	0.551	0.486
	50	100	0.983	0.975	0.884	0.972	0.956	0.846	0.874	0.760	0.672
	50	200	0.992	0.988	0.935	0.986	0.978	0.911	0.931	0.852	0.799
	50	500	0.996	0.994	0.964	0.994	0.989	0.949	0.964	0.903	0.878

Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2, 3, 4, 5$) are independently drawn from the uniform distribution: $U[-1, 1]$. $g_1(x) = sin(2\pi x)$, $g_2(x) = sin(\pi x)$ and $g_3(x) = |cos(\pi x)|$. The factor loading functions are generated as $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ and $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. 3 factors are estimated at each τ using the QFA proposed by Chen et al. (2021), and the reported results are the averages of the adjusted R^2 of regressing the true factors on the estimated factors from 1000 replications.

Table 5: Factor estimation using PCA and PPCA

	T	n	$N(0,1)$ f_{1t}	f_{2t}	f_{3t}	$T(3)$ f_{1t}	f_{2t}	f_{3t}	Cauchy(0,1) f_{1t}	f_{2t}	f_{3t}
PCA	10	50	0.955	0.921	0.420	0.847	0.723	0.455	0.271	0.250	0.392
	10	100	0.964	0.929	0.450	0.858	0.757	0.514	0.286	0.289	0.410
	10	200	0.970	0.944	0.478	0.871	0.751	0.530	0.289	0.285	0.422
	10	500	0.975	0.944	0.493	0.879	0.767	0.568	0.292	0.303	0.433
	50	50	0.973	0.957	0.084	0.894	0.781	0.079	0.003	-0.001	0.032
	50	100	0.986	0.977	0.131	0.937	0.862	0.116	0.032	0.031	0.066
	50	200	0.993	0.988	0.149	0.961	0.901	0.141	0.044	0.048	0.075
	50	500	0.997	0.994	0.166	0.977	0.933	0.161	0.055	0.054	0.091
PPCA	10	50	0.949	0.962	0.382	0.843	0.866	0.379	0.277	0.282	0.387
	10	100	0.989	0.984	0.374	0.960	0.930	0.379	0.321	0.314	0.406
	10	200	0.995	0.993	0.382	0.983	0.969	0.383	0.318	0.309	0.409
	10	500	0.998	0.997	0.400	0.994	0.989	0.402	0.321	0.317	0.417
	50	50	0.953	0.963	0.060	0.858	0.882	0.054	0.003	0.001	0.029
	50	100	0.987	0.982	0.095	0.962	0.947	0.085	0.036	0.031	0.062
	50	200	0.994	0.992	0.110	0.982	0.974	0.100	0.048	0.049	0.072
	50	500	0.998	0.997	0.130	0.994	0.990	0.114	0.058	0.056	0.090

Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2, 3, 4, 5$) are independently drawn from the uniform distribution: $U[-1, 1]$. $g_1(x) = sin(2\pi x),\ g_2(x) = sin(\pi x)\ and\ g_3(x) = |cos(\pi x)|$. The factor loading functions are generated as $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id}),\ \lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ and $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. 3 factors are estimated using PCA and PPCA respectively, and the reported results are the averages of the adjusted R^2 of regressing the true factors on the estimated factors from 1000 replications.

Table 6: Factor estimation using QPCA: $R = 3, D = 2$

	T	n	$N(0,1)$ f_{1t}	f_{2t}	f_{3t}	$T(3)$ f_{1t}	f_{2t}	f_{3t}	Cauchy(0,1) f_{1t}	f_{2t}	f_{3t}
$\tau = 0.25$	10	50	0.731	0.880	0.606	0.588	0.806	0.617	0.348	0.481	0.602
	10	100	0.932	0.948	0.830	0.877	0.902	0.799	0.655	0.668	0.731
	10	200	0.969	0.978	0.916	0.947	0.960	0.901	0.778	0.837	0.847
	10	500	0.990	0.993	0.974	0.983	0.989	0.968	0.929	0.941	0.942
	50	50	0.665	0.875	0.485	0.488	0.782	0.473	0.100	0.202	0.390
	50	100	0.934	0.957	0.811	0.892	0.921	0.766	0.473	0.531	0.602
	50	200	0.969	0.982	0.906	0.949	0.970	0.889	0.785	0.839	0.796
	50	500	0.990	0.993	0.968	0.984	0.989	0.961	0.952	0.963	0.931
$\tau = 0.5$	10	50	0.643	0.889	0.152	0.524	0.837	0.165	0.364	0.664	0.201
	10	100	0.927	0.949	0.127	0.907	0.935	0.136	0.807	0.845	0.171
	10	200	0.968	0.981	0.128	0.955	0.974	0.136	0.917	0.940	0.158
	10	500	0.990	0.994	0.135	0.987	0.991	0.126	0.979	0.986	0.142
	50	50	0.697	0.913	-0.013	0.581	0.870	-0.011	0.279	0.682	0.005
	50	100	0.945	0.968	0.004	0.929	0.956	0.003	0.857	0.899	0.004
	50	200	0.973	0.984	0.011	0.967	0.980	0.012	0.945	0.968	0.014
	50	500	0.991	0.994	0.018	0.989	0.993	0.017	0.984	0.989	0.018
$\tau = 0.75$	10	50	0.718	0.878	0.603	0.609	0.804	0.629	0.356	0.473	0.596
	10	100	0.932	0.948	0.834	0.874	0.900	0.791	0.636	0.664	0.737
	10	200	0.970	0.980	0.922	0.943	0.962	0.907	0.796	0.833	0.848
	10	500	0.991	0.993	0.975	0.984	0.987	0.968	0.933	0.941	0.943
	50	50	0.663	0.872	0.498	0.485	0.779	0.476	0.102	0.203	0.392
	50	100	0.935	0.956	0.813	0.889	0.920	0.762	0.450	0.510	0.608
	50	200	0.969	0.981	0.906	0.951	0.970	0.890	0.792	0.845	0.800
	50	500	0.990	0.993	0.969	0.984	0.989	0.962	0.951	0.964	0.931

Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. The number of characteristics is 2 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1,2$) are independently drawn from uniform distribution: $U[-1,1]$. $g_1(x) = sin(2\pi x),\ g_2(x) = sin(\pi x)\ and\ g_3(x) = |cos(\pi x)|$. $\lambda_{i1} = \sum_{d=1,2} g_1(x_{id}),\ \lambda_{i2} = \sum_{d=1,2} g_2(x_{id})\ and\ \lambda_{i3} = \sum_{d=1,2} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. 3 factors are estimated at each τ using the method proposed in this paper, and the reported results are the averages of the adjusted R^2 of regressing the true factors on the estimated factors from 1000 replications.

Table 7: Factor estimation using SQFA: $R = 3, D = 2$

	T	n	$N(0,1)$ f_{1t}	f_{2t}	f_{3t}	$T(3)$ f_{1t}	f_{2t}	f_{3t}	Cauchy(0,1) f_{1t}	f_{2t}	f_{3t}
$\tau = 0.25$	10	50	0.406	0.826	0.205	0.364	0.793	0.218	0.295	0.649	0.277
	10	100	0.595	0.698	0.195	0.578	0.690	0.208	0.556	0.656	0.231
	10	200	0.623	0.706	0.223	0.604	0.680	0.238	0.573	0.676	0.277
	10	500	0.630	0.682	0.233	0.614	0.687	0.239	0.596	0.686	0.301
	50	50	0.311	0.845	0.040	0.267	0.804	0.058	0.191	0.662	0.101
	50	100	0.516	0.680	0.036	0.503	0.660	0.043	0.471	0.627	0.063
	50	200	0.523	0.691	0.058	0.518	0.676	0.067	0.487	0.649	0.095
	50	500	0.578	0.656	0.061	0.553	0.651	0.068	0.543	0.626	0.102
$\tau = 0.5$	10	50	0.383	0.849	0.133	0.351	0.819	0.132	0.314	0.749	0.142
	10	100	0.584	0.695	0.150	0.573	0.688	0.143	0.531	0.674	0.154
	10	200	0.584	0.713	0.156	0.559	0.716	0.157	0.539	0.678	0.158
	10	500	0.615	0.689	0.157	0.600	0.663	0.157	0.598	0.649	0.152
	50	50	0.277	0.865	-0.014	0.236	0.843	-0.013	0.185	0.773	-0.014
	50	100	0.509	0.679	0.007	0.471	0.669	0.007	0.439	0.625	0.005
	50	200	0.514	0.688	0.015	0.493	0.680	0.016	0.452	0.669	0.016
	50	500	0.557	0.639	0.022	0.544	0.636	0.023	0.503	0.623	0.022
$\tau = 0.75$	10	50	0.402	0.816	0.200	0.374	0.804	0.213	0.316	0.630	0.268
	10	100	0.606	0.688	0.191	0.564	0.900	0.195	0.556	0.666	0.226
	10	200	0.590	0.691	0.221	0.584	0.962	0.215	0.582	0.672	0.281
	10	500	0.638	0.691	0.231	0.622	0.987	0.259	0.620	0.675	0.285
	50	50	0.318	0.837	0.039	0.268	0.779	0.049	0.191	0.658	0.099
	50	100	0.525	0.671	0.039	0.499	0.920	0.044	0.465	0.624	0.067
	50	200	0.528	0.688	0.057	0.510	0.970	0.064	0.481	0.654	0.101
	50	500	0.574	0.652	0.063	0.564	0.989	0.067	0.543	0.627	0.096

Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0, 1)$. The number of characteristics is 2 and all characteristics x_{id} $(i = 1, ...N$ and $d = 1, 2)$ are independently drawn from uniform distribution: $U[-1, 1]$. $g_1(x) = sin(2\pi x),\ g_2(x) = sin(\pi x)\ and\ g_3(x) = |cos(\pi x)|$. $\lambda_{i1} = \sum_{d=1,2} g_1(x_{id}),\ \lambda_{i2} = \sum_{d=1,2} g_2(x_{id})\ and\ \lambda_{i3} = \sum_{d=1,2} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. 2 factors are estimated at each τ using the method proposed by Ma et al. (2021), and the reported results are the averages of the adjusted R^2 of regressing the true factors on the estimated factors from 1000 replications.

Table 8: Estimated numbers of factors

		Five largest eigenvalues of $\hat{Y}\hat{Y}'$					p_n	$\hat{r}$	$\hat{r}_{QFA}$
mean(PPCA)		0.929	0.090	0.081	0.066	0.043		1	
quantile	τ=0.5	0.887	0.094	0.084	0.053	0.043	0.224	1	1
	τ=0.25	1.713	0.110	0.098	0.059	0.047	0.311	1	1
	τ=0.75	2.706	0.115	0.087	0.074	0.067	0.391	1	1
	τ=0.05	8.415	0.311	0.173	0.161	0.138	0.690	1	1
	τ=0.95	13.715	0.567	0.428	0.291	0.246	0.880	1	1

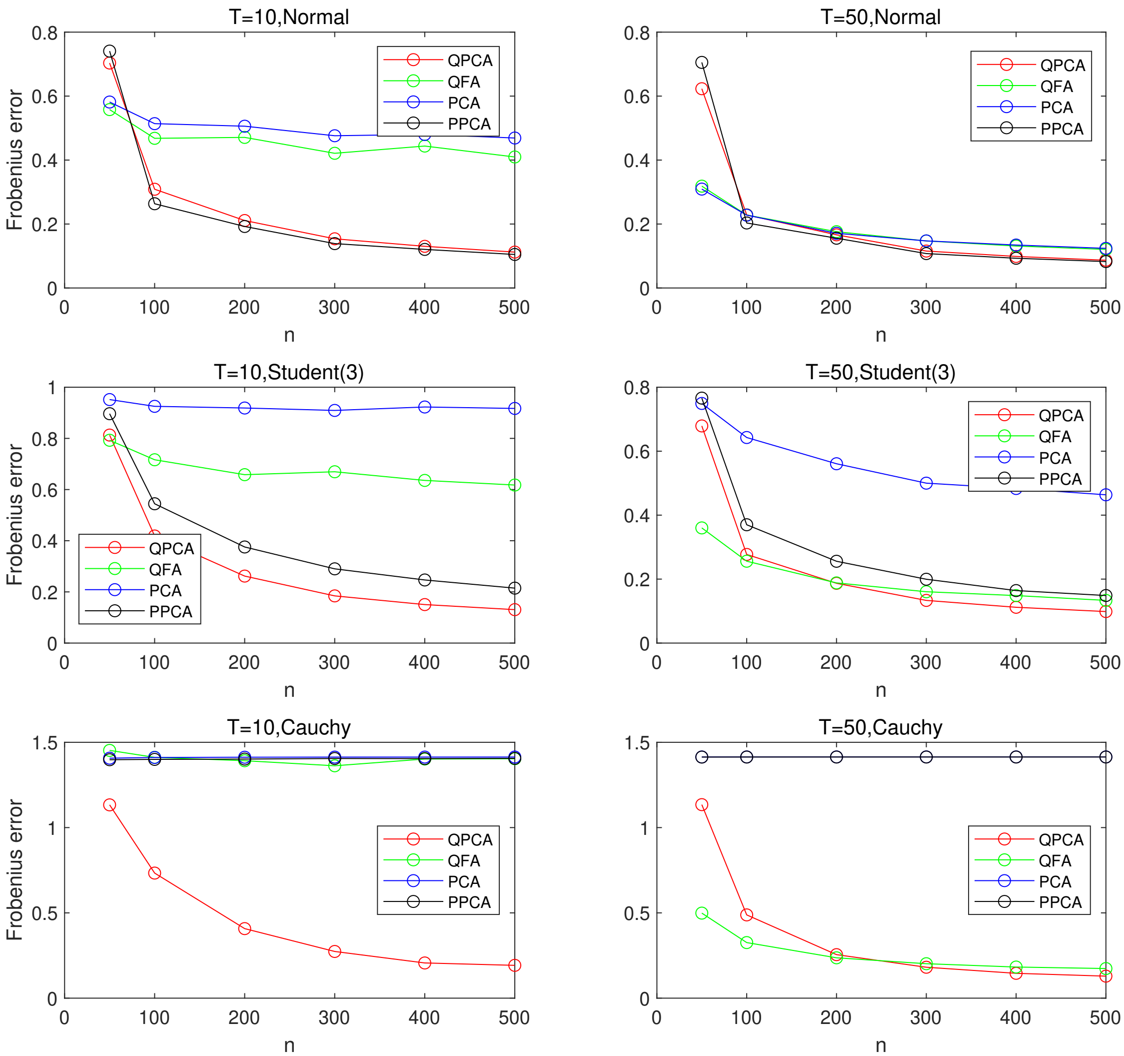
Note: this table shows the estimated numbers of factors using the eigen-ratio estimator proposed by Fan et al. (2016), the proposed estimator in this paper, and the rank-minimization estimator proposed by Chen et al. (2021) for different τs. Column 3 to Column 7 give the 5 largest eigenvalues of $\hat{Y}\hat{Y}'$, where $\hat{Y}$ is the matrix of fitted values in the first-step sieve regressions, and p_n is the threshold value defined in (9).

Table 9: Correlations and means of estimated factors

	τ=0.05	τ=0.25	τ=0.5	τ=0.75	τ=0.95	PPCA	Mean
τ=0.05	1	0.922	0.852	0.767	0.611	0.863	0.943
τ=0.25		1	0.975	0.924	0.753	0.973	0.738
τ=0.5			1	0.971	0.814	0.990	-0.121
τ=0.75				1	0.877	0.979	-0.784
τ=0.95					1	0.862	-0.943
PPCA						1	-0.231

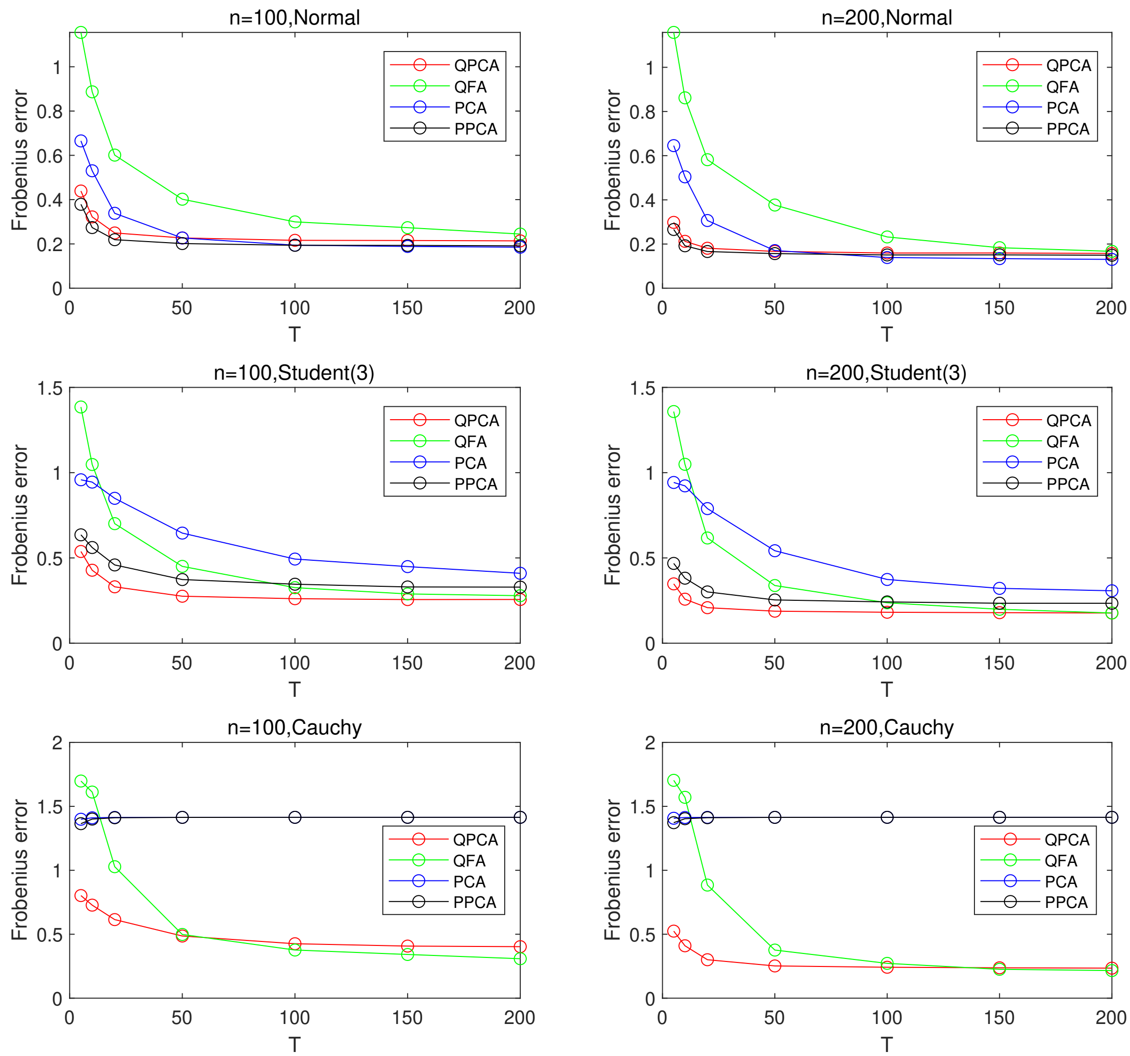
Note: this table shows the correlations and sample means of the estimated mean factor using PPCA and the estimated quantile factors at different τs using QPCA.

Figure 1: Estimation of factors: fixed T and increasing n.



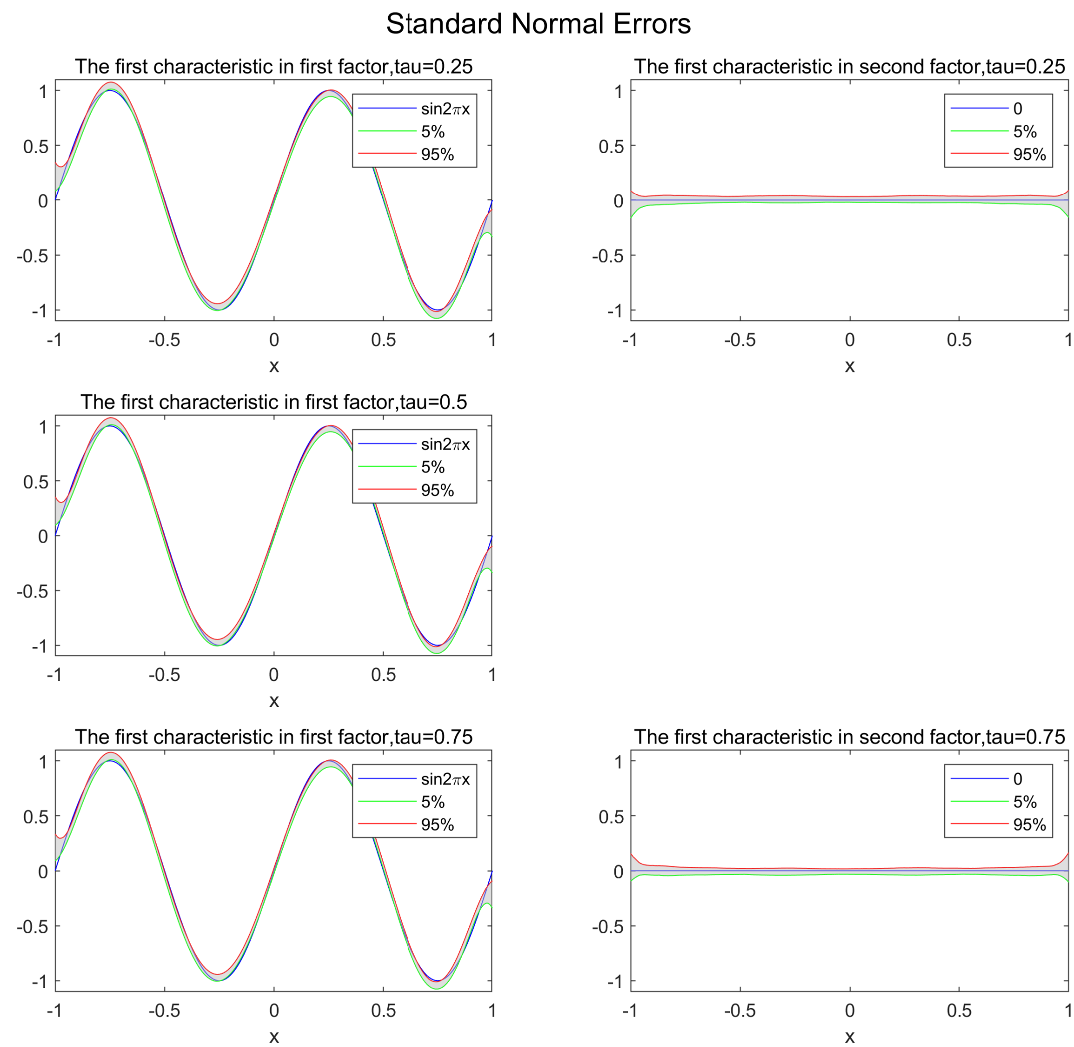
Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2, 3, 4, 5$) are independently drawn from the uniform distribution: $U[-1, 1]$. $g_1(x) = sin(2\pi x)$, $g_2(x) = sin(\pi x)\ and\ g_3(x) = |cos(\pi x)|$. The factor loading functions are generated as $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ and $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. The mean factors (f_{t1} and f_{t2}) are estimated by four methods: PCA, PPCA, QFA and QPCA at $\tau = 0.5$. The reported results are the average Frobenius errors: $\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T}$ from 1000 repetitions, where $\hat{\boldsymbol{H}}$ is the associated rotation matrix for each estimator.

Figure 2: Estimation of factors: fixed n and increasing T.



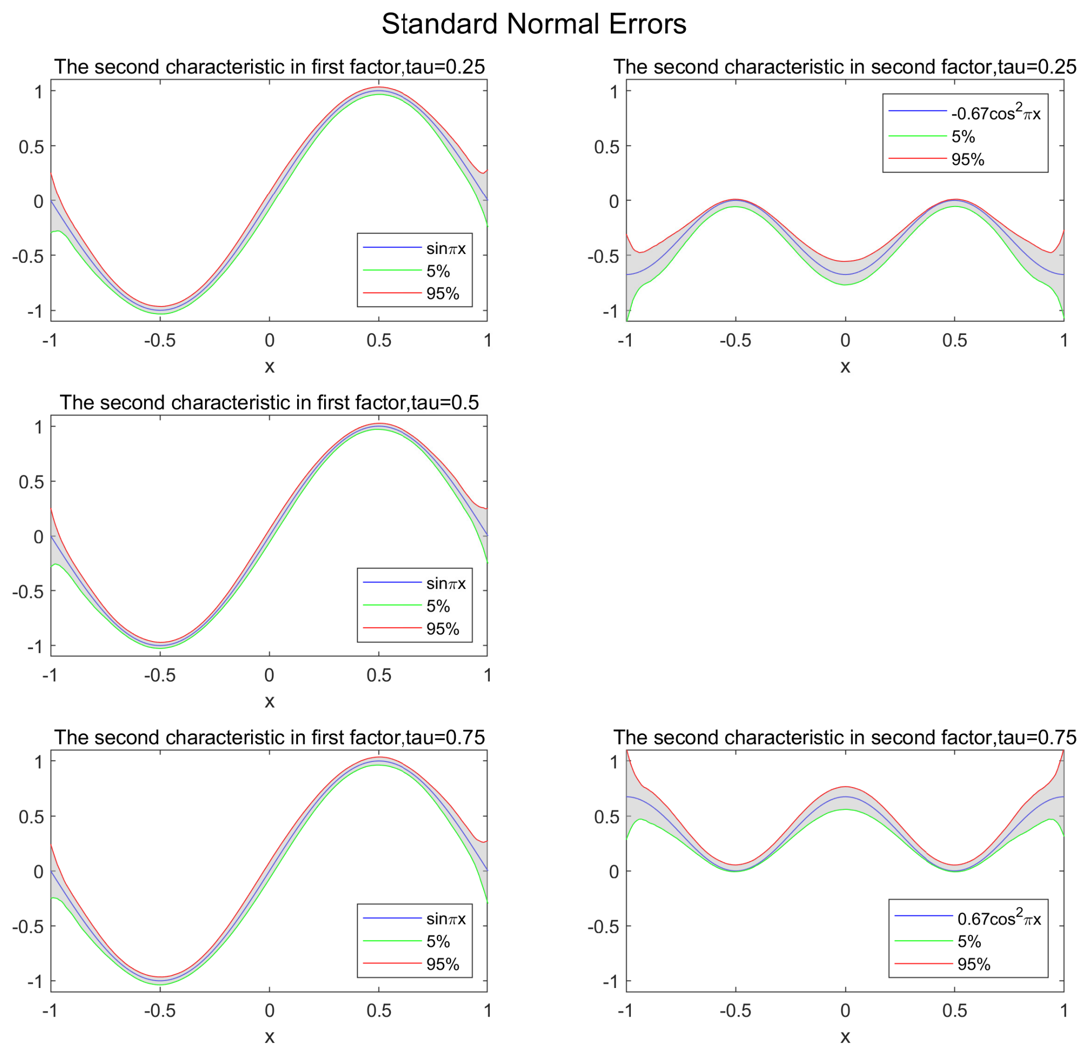
Note: the DGP is $Y_{it} = \lambda_{i1} f_{t1} + \lambda_{i2} f_{t2} + (\lambda_{i3} f_{t3}) u_{it}$, where $f_{t3} = |h_t|, f_{t1}, f_{t2}, h_t \sim i.i.d\ N(0,1)$. The number of characteristics is 5 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2, 3, 4, 5$) are independently drawn from the uniform distribution: $U[-1, 1]$. $g_1(x) = sin(2\pi x)$, $g_2(x) = sin(\pi x)\ and\ g_3(x) = |cos(\pi x)|$. The factor loading functions are generated as $\lambda_{i1} = \sum_{d=1,3,5} g_1(x_{id})$, $\lambda_{i2} = \sum_{d=1,2} g_2(x_{id})$ and $\lambda_{i3} = \sum_{d=3,4} g_3(x_{id})$. $\{u_{it}\}$ are i.i.d draws from three different distributions. The mean factors (f_{t1} and f_{t2}) are estimated by four methods: PCA, PPCA, QFA and QPCA at $\tau = 0.5$. The reported results are the average Frobenius errors: $\|\hat{\boldsymbol{F}} - \boldsymbol{F}\hat{\boldsymbol{H}}\|/\sqrt{T}$ from 1000 repetitions, where $\hat{\boldsymbol{H}}$ is the associated rotation matrix for each estimator.

Figure 3: Loading function of the first characteristic when error term is normal



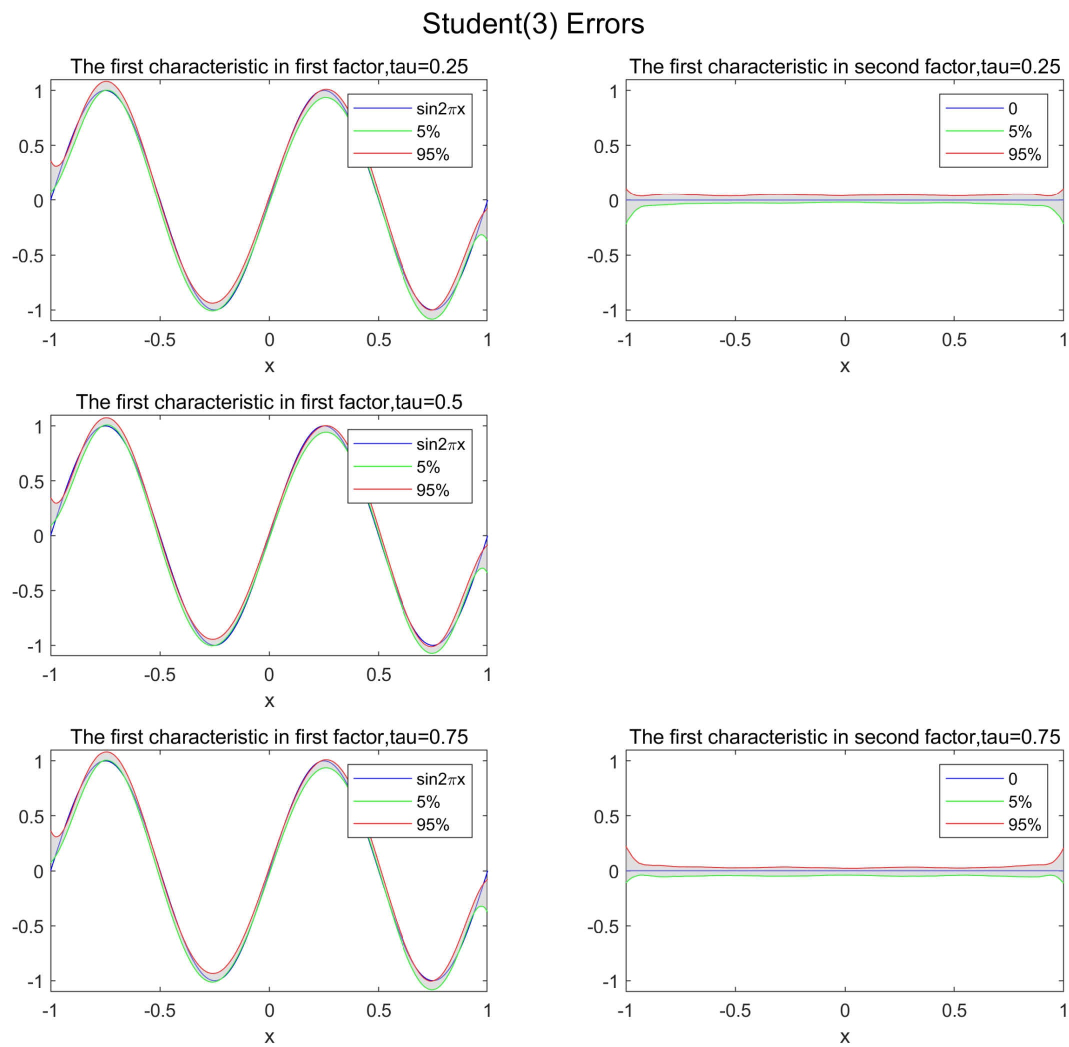
Note: the DGP is: $y_{it} = \lambda_{i1} f_{t1} + (\lambda_{i2} f_{t2}) u_{it}$, where $f_{t2} = |g_t|$ and $f_{t1}, g_t \sim i.i.d\ N(0, 1)$. $n = 500, T = 10$. The number of characteristics is 2 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2$) are independently drawn from uniform distribution: $U[-1, 1]$. $g_{11}(x) = sin(2\pi x), g_{21}(x) = 0, g_{12}(x) = sin(\pi x)$, $g_{22}(x) = cos^2(\pi x)$, and $\lambda_{i1} = g_{11}(x_{i1}) + g_{12}(x_{i2}), \lambda_{i2} = g_{21}(x_{i1}) + g_{22}(x_{i2})$. u_{it} are drawn independently from the standard normal distribution. The left panel are the estimation results for $g_{11,\tau}(x) = \sin(2\pi x)$ and the right panel are the estimation results for $g_{21,\tau}(x) = 0$ with $\tau \in \{0.25, 0.75\}$. For each graph, the blue line is the true function, the red line and the green line are the 95% and 5% empirical quantiles from 1000 replications.

Figure 4: Loading function of the second characteristic when error term is normal



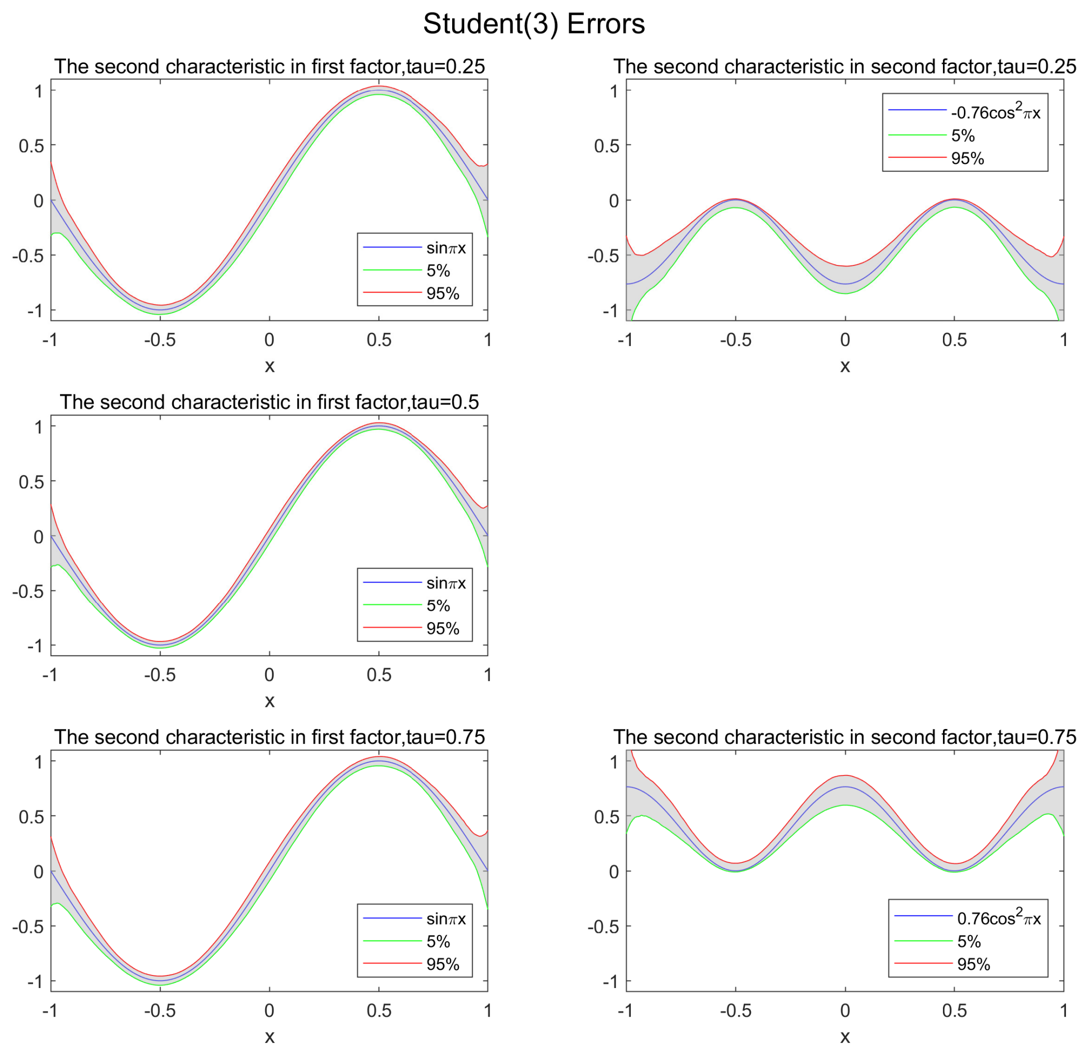
Note: the DGP is: $y_{it} = \lambda_{1i} f_{1t} + (\lambda_{2i} f_{2t}) u_{it}$, where $f_{2t} = |g_t|$ and $f_{1t}, g_t \sim i.i.d\ N(0, 1)$. $n = 500, T = 10$. The number of characteristics is 2 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2$) are independently drawn from uniform distribution: $U[-1, 1]$. $g_{11}(x) = sin(2\pi x), g_{21}(x) = 0, g_{12}(x) = sin(\pi x)$, $g_{22}(x) = cos^2(\pi x)$, and $\lambda_{1i} = g_{11}(x_{1i}) + g_{12}(x_{2i}), \lambda_{2i} = g_{21}(x_{1i}) + g_{22}(x_{2i})$. u_{it} are drawn independently from the standard normal distribution. The left panel are the estimation results for $g_{12,\tau}(x) = \sin(\pi x)$ and the right panel are the estimation results for $g_{22,\tau}(x)$ with $\tau \in \{0.25, 0.75\}$. For each graph, the blue line is the true function, the red line and the green line are the 95% and 5% empirical quantiles from 1000 replications.

Figure 5: Loading function of first characteristic when error term is $T(3)$



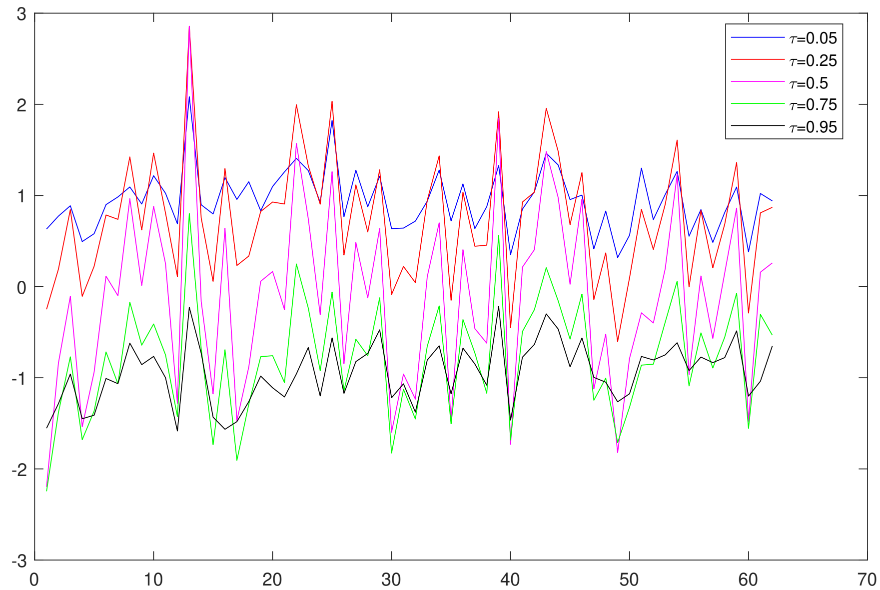
Note: the DGP is: $y_{it} = \lambda_{i1} f_{t1} + (\lambda_{i2} f_{t2}) u_{it}$, where $f_{t2} = |g_t|$ and $f_{t1}, g_t \sim i.i.d\ N(0,1)$. $n = 500, T = 10$. The number of characteristics is 2 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2$) are independently drawn from uniform distribution: $U[-1, 1]$. $g_{11}(x) = sin(2\pi x), g_{21}(x) = 0, g_{12}(x) = sin(\pi x)$, $g_{22}(x) = cos^2(\pi x)$, and $\lambda_{i1} = g_{11}(x_{i1}) + g_{12}(x_{i2}), \lambda_{i2} = g_{21}(x_{i1}) + g_{22}(x_{i2})$. u_{it} are drawn independently from the student's t distribution with 3 degrees of freedom. The left panel are the estimation results for $g_{11,\tau}(x) = \sin(2\pi x)$ and the right panel are the estimation results for $g_{21,\tau}(x) = 0$ with $\tau \in \{0.25, 0.75\}$. For each graph, the blue line is the true function, the red line and the green line are the 95% and 5% empirical quantiles from 1000 replications.

Figure 6: Loading function of second characteristic when error term is $T(3)$



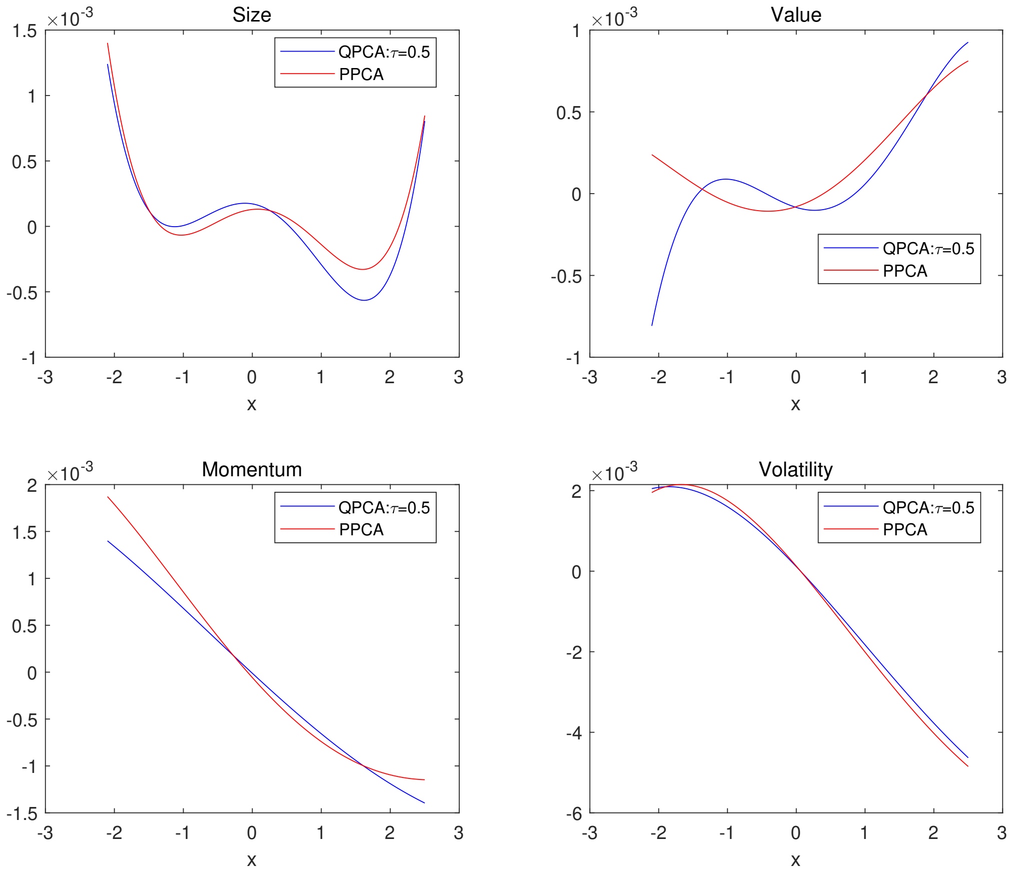
Note: the DGP is: $y_{it} = \lambda_{i1} f_{t1} + (\lambda_{i2} f_{t2}) u_{it}$, where $f_{t2} = |g_t|$ and $f_{t1}, g_t \sim i.i.d\ N(0,1)$. $n = 500, T = 10$. The number of characteristics is 2 and all characteristics x_{id} ($i = 1, ...N$ and $d = 1, 2$) are independently drawn from uniform distribution: $U[-1, 1]$. $g_{11}(x) = sin(2\pi x), g_{21}(x) = 0, g_{12}(x) = sin(\pi x)$, $g_{22}(x) = cos^2(\pi x)$, and $\lambda_{i1} = g_{11}(x_{i1}) + g_{12}(x_{i2}), \lambda_{i2} = g_{21}(x_{i1}) + g_{22}(x_{i2})$. u_{it} are drawn independently from the student's t distribution with 3 degrees of freedom. The left panel are the estimation results for $g_{12,\tau}(x) = \sin(\pi x)$ and the right panel are the estimation results for $g_{22,\tau}(x)$ with $\tau \in \{0.25, 0.75\}$. For each graph, the blue line is the true function, the red line and the green line are the 95% and 5% empirical quantiles from 1000 replications.

Figure 7: Estimated loading functions using QPCA for different τs



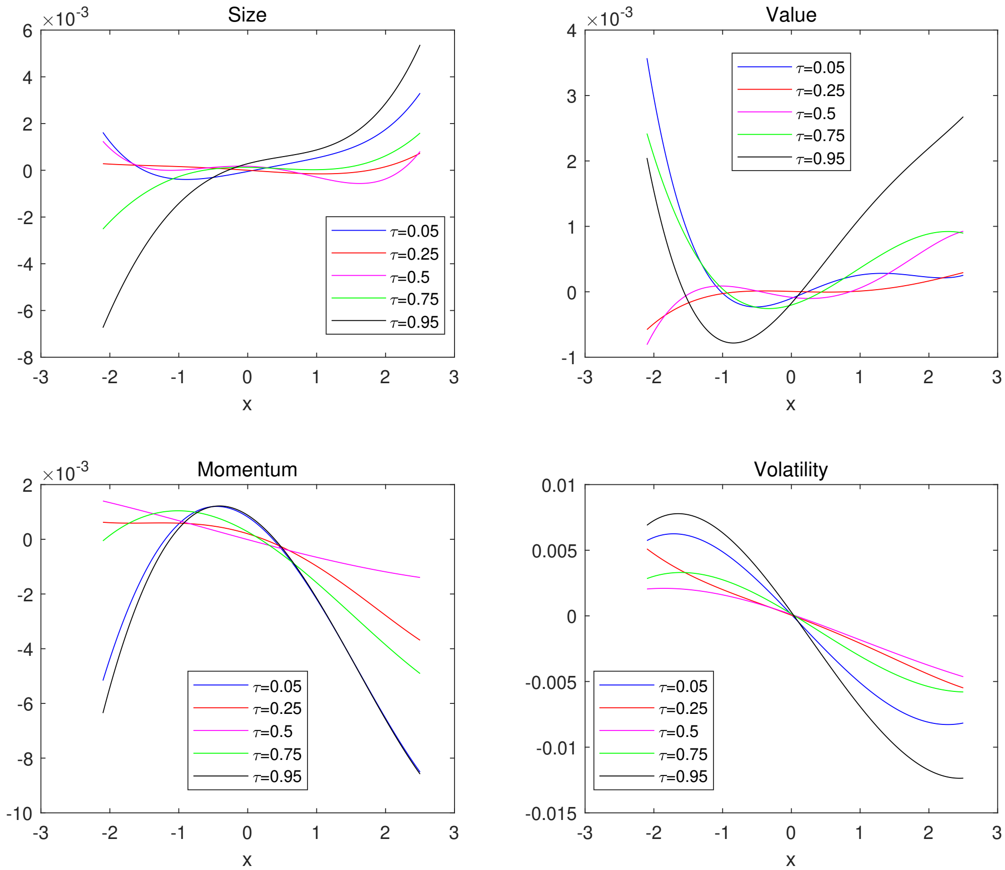
Note: this figure plots the estimated quantile factors at different quantiles using the proposed estimation method.

Figure 8: Estimated loading functions using PPCA and QPCA for $\tau = 0.5$



Note: this figure plots the estimated quantile factor loading functions of the four characteristics using PPCA and QPCA at $\tau = 0.5$

Figure 9: Estimated loading functions using QPCA for different τs



Note: this figure plots the estimated quantile factor loading functions of the four characteristics using QPCA at different τs.

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